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ABSTRACT

This is part two of a two-part SMSG mathematics text for junior high school students. Key ideas emphasized are structure of arithmetic from an algebraic viewpoint, the real number system as a progressing development, and metric and non-metric relations in geometry. Chapter topics include the rational number system; parallels, parallelograms, triangles, and right prisms; circles; statistics and graphs; mathematical systems; and mathematics at work in science. Slight revisions are contained in a later edition. (MP)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

VOLUME I (Part 2)

2

SE 027 920

MATHEMATICS FOR JUNIOR HIGH SCHOOL

Volume I (Part 2)

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CHAPTER 8

THE RATIONAL NUMBER SYSTEM II

8-1. Ratios and Proportion

Sometimes we can discover interesting things by comparing different numbers. For example, consider the following situation.

One sunny day a boy measured the length of the shadow cast by each member of his family. He also measured the length of the shadow cast by a big tree in their yard. He found that his father, who is 72 inches tall, cast a shadow 48 inches long. His mother, who is 63 inches tall, cast a shadow 42 inches long. His little brother, who is only 30 inches high, cast a shadow 20 inches long. He didn't know how tall the tree was, but its shadow was 40 feet long.

Let us arrange this information in a table.

	Shadow length	Height
Father	48 inches	72 inches
Mother	42 inches	63 inches
Brother	20 inches	30 inches
Tree	40 feet	?

We see that the taller people have longer shadows. But let us examine this more closely. Suppose we divide the shadow length of the little brother by his height. We get $\frac{20}{30}$ or $\frac{2}{3}$. Suppose we try the same thing for the father. It will be easier to measure the father's height and shadow in feet. The father is 6 feet tall and his shadow is 4 feet long. If we divide shadow length by height we get $\frac{4}{6}$ or $\frac{2}{3}$.

Now do this for the mother--divide her shadow length by her height. Do you again get $\frac{2}{3}$?

Let us assume that this principle holds for all objects (we must measure the shadows at the same time; the shadow changes during the day as the position of the sun changes). Then the tree must be 60 feet tall in order that its shadow length divided by its height be $\frac{2}{3}$. Thus we can discover how tall the tree is without actually measuring it!

Definition. The ratio of a number a to a number b $(b \neq 0)$, is the quotient $\frac{a}{b}$. Sometimes this is written a:b.

Thus we have formed the ratio of shadow length to height, and we discovered that this ratio was the same for all the people whom we measured. Using this we were able to discover that the tree was 60 feet tall.

Suppose that the boy's uncle is 66 inches tall (5 feet 6 inches). How long would his shadow be if it were measured at the same time and place as the other people?

To answer this question, we let s be his shadow length measured in inches. Then we must have:

$$\frac{s}{66} = \frac{2}{3}.$$

Since $\frac{s}{66}$ and $\frac{2}{3}$ are the same number we find that:

$$\begin{aligned} 66\left(\frac{s}{66}\right) &= 66\left(\frac{2}{3}\right) \\ s &= \frac{2}{3} \times 66 \\ &= 2 \times 22 \\ &= 44. \end{aligned}$$

So his shadow would be 44 inches long.

Exercises 8-1a

1. What is your height in inches? What would be the length of your shadow if it were measured at the same time and place as the people in our story?

2. Some other objects were measured at another time and place, and the data are recorded below. Fill our the table completely.

	Shadow length	Height	Ratio of shadow length to height
Garage	12 feet	27 feet	
Clothes pole	36 inches		$\frac{4}{9}$
Tree	10 feet	$22\frac{1}{2}$ feet	
Flag pole		144 inches	$\frac{4}{9}$
Fence	$13\frac{1}{3}$ inches	30 inches	

3. In a class there are 36 children of whom 12 are girls.
   - (a) How many boys are in the class?
   - (b) What is the ratio of the number of girls to the total number of children in the class?
   - (c) What is the ratio of the number of boys to the total number of children in the class?
   - (d) What is the ratio of the number of girls to the number of boys?

4. In another class, the ratio of the number of girls to the number of boys is the same as in the previous class. This class has 36 girls. How many boys does it have?

5. In a third class, in which the ratio of the number of girls to the number of boys is the same, there are 100 students. Is such a class possible? Why? How many boys would there have to be?

6. In Chapter 6 you learned that $\frac{a}{b} = \frac{c}{c}$ when ad = bc. State which of the following ratios are equal.

(a) $\frac{10}{5}, \frac{20}{10}$

(b) $\frac{6}{3}, \frac{16}{9}$

(c) $\frac{48}{16}, \frac{42}{14}$

(d) $\frac{68}{17}, \frac{76}{19}$

(e) $\frac{15}{13}, \frac{45}{39}$

7. In each of the following, determine x so that the equality between the ratios is valid:

(a) $\frac{20}{8} = \frac{x}{6}$

(b) $\frac{14}{30} = \frac{x}{90}$

(c) $\frac{x}{3} = \frac{75}{15}$

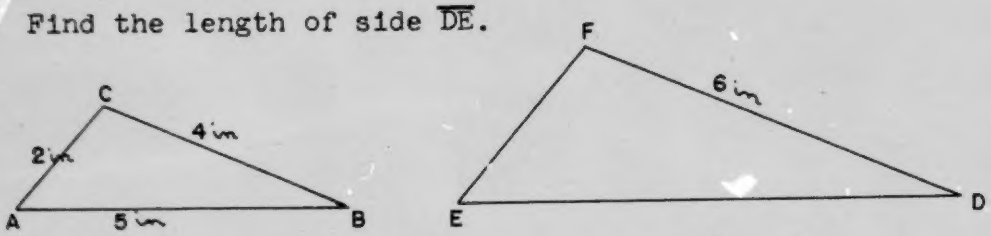
(d) $\frac{81}{108} = \frac{x}{12}$

(e) $\frac{x}{42} = \frac{36}{27}$

*8. When triangles are the same shape, but not the same size, they are called similar triangles. The measures of the lengths of corresponding sides of similar triangles form equal ratios. Triangles ABC and DEF are similar.

(a) Find the length of side $\overline{EF}$.

(b) Find the length of side $\overline{DE}$.



9. A cookie recipe calls for the following items.

1 cup butter

$\frac{2}{3}$ cup sugar

2 eggs

$1\frac{1}{2}$ cups flour

1 teaspoon vanilla

This recipe will make 60 cookies. Rewrite the recipe by enlarging it in the ratio 3:1. How many cookies will it make now?

Suppose you wanted to make only 30 cookies--how much would you need of each of the items listed above?

10. Joyce has a picture negative 4 inches wide and 5 inches long. She wants an enlargement that will be 8 inches wide. How long will the enlarged print be?

11. If a water tower casts a shadow 275 feet long and a 6 foot man casts a shadow 15 feet long, how tall is the water tower?

12. Mr. Landry was paid $135 for a job which required 40 hours of work. At this rate, how much would he be paid for a job that required 60 hours?

Taking another look at our example and at the definition of ratio, we see that a ratio compares just two numbers. In our example we had two sets of numbers, the lengths of the shadows, and the heights. In each case we formed the ratio of the first number (the shadow length) to the second number (the height). In our example these ratios were all equal to one another. In such a situation, we say that the physical quantities measured by the numbers are proportional to one another.

Definition. Two physical quantities are said to be proportional to one another if the ratio of their measures is always the same. The value of this ratio is called the constant of proportionality. A proportion is a statement of equality of two ratios.

Thus in our example, the shadow length is proportional to the height, and the constant of proportionality is $\frac{2}{3}$. If the shadow length of a water tower is 150 in. long and w is the number of inches in the height of the tower then the statement

$$\frac{150}{w} = \frac{2}{3}$$

is a proportion. Another way to define proportional physical quantities is to say that the ratios of any two corresponding values may be used to form a proportion.

Another common example is the physical law which states that the distance traveled at a constant speed is proportional to the time. Let us consider this law in the case of an inaccurate speedometer.

Richard and his parents are setting out for a drive in the desert. While his father drives, Richard watches the instrument panel of the car. He pays special attention to the speedometer, which tells how fast the car is going, to the odometer, which tells how far the car has gone, and to the clock. The speedometer shows a steady speed of 60 miles an hour, with no slowing down or speeding up. He writes down some data as the drive continues.

Table I

Odometer	Clock
34,700	8:00
34,720	8:24
34,725	8:30
34,733.3	8:40.

Richard builds a new table from these data by subtracting each entry from the one below it.

Table II

Distance traveled (miles)	Time elapsed (minutes)	Ratio of distance to time
20	24	$\frac{5}{6}$
5	6	$\frac{5}{6}$
8.3	10	

The first two ratios are both $\frac{5}{6}$, and the last one is almost equal to $\frac{5}{6}$. Indeed, Richard recalls that when he made that reading the tenth's wheel on the odometer had just passed the "3" mark--the entry in Table II really should have been 8 and $\frac{1}{3}$ tenths, which is $8 + \frac{\frac{1}{3}}{10}$. Therefore the ratio should again be $\frac{5}{6}$. So Richard concludes that all these ratios are equal, and thus that the car is traveling at a constant speed. What is that constant speed? To figure this out, let us imagine that the next readings were taken at 9:00, when the elapsed time was 60 minutes or 1 hour. Make this entry in Tables I and II. Assuming a constant speed, the ratio of the distance traveled to the time elapsed would be $\frac{5}{6}$. What should the "distance traveled" entry be in Table II? What should the odometer reading be in Table I?

If your arithmetic is correct, you have found that the "distance traveled" entry in Table II should be 50 miles. But this means that the car's speed is 50 miles per hour, not 60 as the speedometer reads. Richard tells his father that his data do not agree with the speedometer.

When Richard points out to his father that there is an error somewhere, his father says, "Yes, the speedometer is wrong. In fact, the error in the speedometer reading is proportional to the speed it actually reads. Thus, the error at low speeds is small, while it is great at high speeds. The ratio of $\frac{5}{6}$ should enable you to determine the actual speed at any reading of the speedometer." Richard thinks a minute and says, "Yes, when the speedometer reads 30, the car is really going 25." Is Richard correct?

You have seen that a ratio is a comparison of two numbers by division. The numbers may be measures of physical quantities. The word, per, indicates division. It expresses the ratio of the measures of two physical quantities, such as miles and hours. Store prices provide additional examples. Prices relate value to amount such as $1.00 per pound, or $.59 per dozen. In each case, the per indicates a ratio, generally between two different kinds of quantities. This is sometimes called a rate. Notice further that the second quantity in each case represents the standard of comparison. A store charges "10¢ per comb"; one comb is the standard of comparison, and they want 10¢ in the cash register for each comb sold. Of course, this standard does not always represent a quantity of one. For example, cents per dozen, dollars per pair (for shoes), dollars per 1000 (for bricks).

Exercises 8-1b

1. Change each of the time intervals to hours in Table II, and find the ratio of miles to hours in each. Are all the ratios equal? Do you see now that speed is measured by the ratio of the distance traveled to the corresponding time elapsed? Your car speedometer computes this ratio constantly while the car is moving. The units used in its computation are miles and hours, so we call the ratio miles per hour, or m.p.h.

2. See if you can use the above ideas to complete the following table:

	Distance	Time	Speed
(a)	25 miles	____ hours	50 mph
(b)	____ miles	45 minutes	30 mph
(c)	1 mile	45 seconds	____ mph

3. In the case of the car whose speedometer registers 60 m.p.h. when the car actually is going 50 m.p.h., find the actual speed of the car when the speedometer reads

   (a) 15 m.p.h. (b) 45 m.p.h. (c) 0 m.p.h. (d) 80 m.p.

4. (a) What is the cost of 3 dozen doughnuts at $.55 per dozen?

   (b) What is the cost of 12 candy bars at 4 for 15¢? (How could you state this price using the word "per"?)

   (c) What is the cost of 8500 bricks at $14 per thousand?

   (d) A road has a grade of 6%, which means that it rises 6 feet per 100 feet of road. How much does it rise in a mile? Find the answer to the nearest foot.

*5. Another common unit to express speed is "feet per second". A "go-cart" can go 30 m.p.h. How many feet per second is this? (The degree of hazard of speed under various conditions often can be realized better when the speed is given in feet per second).

*6. (Research) The speed of jet aircraft is often given in "Mach" numbers. What do they mean? Do they indicate a ratio? If so, what is the standard of comparison?

7. The following table lists the measures, A and B, of two quantities which are proportional to each other. The first

entry can be used to determine the constant of proportionality. Complete the table:

	A	B	Ratio $\frac{A}{B}$	Ratio in simplest form
(a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)		21		
(c)	30			
(d)		100		
(e)	100			

*8. The area of a triangle is proportional to the product of the base and the height. Find the constant of proportionality.

Multiple ratios

Sometimes it is natural and desirable to compare more than just two quantities. For example, a mixture of nuts calls for 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans. Here the ratio of the number of pounds of peanuts to the number of pounds of cashews is 5:2, and the ratio of the number of pounds of cashews to the number of pounds of pecans is 2:1. This may be stated briefly as 5:2:1. Suppose a grocer wants to put up 24 pounds of this mixture. How many pounds of peanuts, cashews and pecans should he use? To determine your answer, first answer these questions:

(1) If 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans are mixed together, how many pounds of nuts are there?

(2) What is the ratio of the number of pounds of peanuts to the total number of pounds?

(3) Since this ratio will be the same in the mixture whose total weight is 24 pounds, how many pounds of peanuts are required?

(4) Answer questions (2) and (3) when "peanuts" are replaced by "cashews".

(5) Answer questions (2) and (3) when "peanuts" are replaced by "pecans".

*Exercises 8-1c

1. If a nut mixture, with the same ingredients in the same ratio as above, is to total 56 pounds, how many pounds of each kind of nuts will the grocer need?
2. If a nut mixture with the same ingredients, but in the ratio 5:3:2 is to total 100 pounds, how many pounds of each kind of nuts will the grocer need?
3. If a nut mixture with the same ingredients, in ratio 3:2:1, has 30 pound of peanuts, how many pounds of cashews and pecans will be needed? What will the total weight be?
4. A triangle has sides of length 11 inches, 8 inches, and 6 inches. If another triangle, the measures of whose sides have the same ratio, is to be drawn with a perimeter of 100 inches, how long will the shortest side be?

8-2. Percent

Many of you are familiar with the word "percent", and you may know something about its meaning. If your teacher says "90 percent of the answers on this paper are correct", would you know what he means? The word "percent" comes from the Latin phrase "per centum", which means "by the hundred". If the paper with 90% of the answers correct has 100 answers, then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ could be used instead of the phrase "90 percent" to describe the proportion of the answers which are correct. The word PERCENT is used when a ratio is expressed with a denominator of 100.

$$90 \text{ percent} = \frac{90}{100} = 90 \times \frac{1}{100}$$

For convenience a symbol, %, is used for the word "percent". This symbol is a short way of saying $\frac{1}{100}$.

$$\frac{90}{100} = 90 \times \frac{1}{100} = 90\%$$

$$\frac{16}{100} = 16 \times \frac{1}{100} = 16\%$$

$$\frac{37}{100} = \quad ? \quad = 37\%$$

$$\frac{7}{100} = 7 \times \frac{1}{100} = \ ?$$

$$? = 13 \times \frac{1}{100} = 13\%$$

In the case of the paper with 100 problems, 90% of which are answered correctly, what do we know about the other 10 answers? Are you sure that one of the following statements must be true?

Ten problems have incorrect answers.

No solutions are given for any of the ten problems.

Some of the ten problems have solutions with incorrect answers and the remainder have no solutions given.

The ten problems are $\frac{10}{100}$ or 10% of the 100 problems.

Suppose that the paper has 90 correct answers out of the 100; 6 incorrect answers out of the 100; 4 answers omitted out of the

100. Since we know the total number of problems we can express the number of each type of answer in terms of percent.

$$\frac{90}{100} = 90 \times \frac{1}{100} = 90\% \quad \text{(correct answers for 90\% of problems)}$$

$$\frac{6}{100} = 6 \times \frac{1}{100} = 6\% \quad \text{(incorrect answers for 6\% of problems)}$$

$$\frac{4}{100} = 4 \times \frac{1}{100} = 4\% \quad \text{(4\% of problems omitted)}$$

$$\frac{90}{100} + \frac{6}{100} + \frac{4}{100} = \frac{100}{100} \quad \text{(all possible answers)}$$

$$90\% + 6\% + 4\% = 100\% \quad \text{(all possible answers)}$$

100 is another name for the number one.

$$1 = \frac{3}{3} = \frac{13}{13} = \frac{25}{25} = \frac{100}{100} = 100\%$$

The number 2 can be written

$$\frac{2}{1} = \frac{6}{3} = \frac{26}{13} = \frac{50}{25} = \frac{200}{100} = 200\%$$

In other words 200 % means $200 \times \frac{1}{100} = \frac{200}{100} = 2$

The ratio a:b is another way of expressing the number $\frac{a}{b}$. You know that the number $\frac{a}{b}$ can be written in many different ways. The number $\frac{1}{2}$ can be written as $\frac{2}{4}, \frac{8}{16}, \frac{35}{70}, \frac{90}{180}, \frac{50}{100}$. Thus

$$\frac{50}{100} = 50 \times \frac{1}{100} = 50\%$$

Hence 50 % is one way to name the number $\frac{1}{2}$. How can the number $\frac{1}{5}$ be written with the symbol %?

$$\frac{1}{5} = \frac{20}{100} = 20 \times \frac{1}{100} = 20\%$$

A class of 25 pupils is made up of 11 girls and 14 boys. The ratio of the number of girls to the number of pupils in the class can be expressed many ways. For instance:

$$\frac{11}{25} = \frac{22}{50} = \frac{33}{75} = \frac{44}{100} = \frac{55}{125} = \frac{66}{150}$$

If we wish to indicate the percent of the class that is girls, which fraction gives the information most easily? Why? The ratio of the number of boys to the total number in the class may be written

$$\frac{14}{25} = \frac{c}{50} = \frac{d}{75} = \frac{56}{100} = \frac{e}{125} = \frac{f}{150} .$$

What numerators are represented by the letters c , d , e , f? Notice the two ratios $\frac{11}{25}$ (girls) and $\frac{14}{25}$ (boys). What is the sum of the two ratios? Find the sum of the two ratios $\frac{44}{100}$ and $\frac{56}{100}$. Express the two ratios and their sum as percents using the symbol, %. The number of students in the entire class may be expressed as the ratio $\frac{25}{25}$ or the ratio $\frac{100}{100}$ or 100 .

Property 1. Any fraction $\frac{a}{b}$ can be expressed as a percent by finding the number c such that $\frac{a}{b} = \frac{c}{100} = c \times \frac{1}{100} = c\%$

Exercises 8-2a

1. Using squared paper draw a large square whose interior is divided into 100 small squares. Write the letter A in 10 small squares. Write the letter B in 20% of the squares. Write the letter C in 35% of the squares. Write the letter D in 30 of the squares. Write the letter X in the remainder of the squares.
   - (a) In what fraction of the squares is the letter A?
   - (b) In what percent of the squares is the letter A?
   - (c) In how many squares is the letter B?
   - (d) In what fraction (simplest form) of the squares is the letter B?
   - (e) In how many squares is the letter C?

(f) In what fraction (simplest form) of the squares is the letter C?

(g) In what fraction of the squares is the letter D?

(h) In what percent of the squares is the letter D?

(i) In what fraction of the squares is the letter X?

(j) In what percent of the squares is the letter X?

2. What is the sum of the fractions in parts (a), (d), (f), (g), (i), of problem 1?

3. What is the sum of the percents of the squares containing the letters A, B, C, D, X?

4. Write each of the following numbers as percent.

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{1}{5}$ (e) $\frac{2}{5}$ (f) $\frac{3}{5}$

(g) $\frac{4}{5}$ (h) $\frac{2}{2}$ (i) $\frac{4}{4}$ (j) $\frac{7}{5}$

5. Jean has a weekly allowance of 50 cents. One week she spends 12 cents for a pencil, 10 cents for an ice cream cone, 15 cents for Sunday school collection, and puts the rest in her piggy bank.

(a) Express each amount as a ratio of the whole, and also as percent.

(b) Find the sum of the ratios.

(c) Find the sum of the percents.

(d) What check on the separate answers above do you observe?

6. The monthly income for a family is \$400. The family budget for the month is shown.

Payment on the mortgage for the home	\$80
Taxes	20

Payment on the car	36
Food	120
Clothing	48
Operating expenses	32
Health, Recreation, etc.	24
Savings, and Insurance	40

(a) What percent of the income is assigned to each item of the budget?

(b) What is one check on the accuracy of the 8 answers?

Percent is a convenient tool for giving information involving ratios. Athletic standings usually are given in percent. Two seventh grade pupils discovered the reason for this use of percent. The boys were discussing the scores of their baseball teams. In the Little League one team won 15 games out of 20 games played. Another team won 18 out of 25 games. Which team had a better record? The second team won 3 more games, but the first team played fewer games. Look at the ratios of the number of games won to the number of games played for each team. The ratios $\frac{15}{20}$ and $\frac{18}{25}$ cannot be compared at a glance. Let us use percent for the comparison.

The first team won $\frac{15}{20}$ of the games played.

$\frac{15}{20} = \frac{75}{100} = 75\%$. They won 75% of the games played.

The second team won $\frac{18}{25}$ of the games played.

$\frac{18}{25} = \frac{72}{100} = 72\%$ They won 72% of the games played.

The first team which won 75% of its games had a higher standing than the second team which won 72% of its games. We could say that $72\% < 75\%$, or $75\% > 72\%$.

Data about business, school, Boy Scouts--are often given in percent. It is more convenient to refer to the data at some later time if it is given in percent than if it is given otherwise.

A few years ago the director of a boy scout camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

(1) There were 200 boys in camp.

(2) One hundred percent of the boys were hungry for the first dinner in camp.

(3) Fifteen percent of the boys forgot to pack a toothbrush, and needed to buy one at camp.

(4) On the second day in camp 44 boys caught fish.

(5) One boy wanted to go home the first night.

(6) A neighboring camp director said "Forty percent of the boys in my camp will learn to swim this summer. We shall teach 32 boys to swim."

From items 1 and 2, how many hungry boys came to dinner the first day?

$$100 \quad = \frac{100}{100} \times 200 = 200$$

Of course we should know without computation that 100% of 200 is 200.

From item 3, how many extra toothbrushes were needed?

$$15\% \text{ means } 15 \times \frac{1}{100} = \frac{15}{100}$$

$$\frac{15}{100} \times 200 = \frac{3000}{100} = 30.$$

Another way to do it is to find the number x such that

$$\frac{15}{100} = \frac{x}{200}$$

$$\frac{15}{100} = \frac{30}{200}$$

So $x = 30$, and 30 toothbrushes are needed.

From item 4, we can compute the percent of the number of boys who caught fish on the second day. We wish to find x such that

$$\frac{44}{200} = \frac{x}{100}$$

$$\frac{44}{200} = \frac{22}{100}$$

$$\frac{22}{100} = 22 \times \frac{1}{100} = 22\%$$

Therefore 22% of the boys caught fish on the second day.

From item 5, we can compute the percent of the total number of boys who were homesick. We wish to find x such that

$$\frac{1}{200} = \frac{x}{100}$$

$$\frac{1}{200} = \frac{\frac{1}{2}}{100}$$

$$\frac{\frac{1}{2}}{100} = \frac{1}{2} \times \frac{1}{100} = \frac{1}{2}\%$$

Another way to say "$\frac{1}{2}\%$" is to say "$\frac{1}{2}$ of 1%".

1% of 200 means $\frac{1}{100} \times 200 = 2$

So 2 boys are 1% of the total number of boys, and 1 boy is $\frac{1}{2}$ of 1% of the number of boys. Sometimes it is convenient to use 1% of a number and 10% of the number in solving problems. In the case of the 200 boy scouts, 1% of the 200 is 2, as shown above. Ten percent of them would be

$$\frac{10}{100} \times 200 = 20 .$$

In determining how many boys are 15% of the group (those who forgot toothbrushes), we might say that 15% is 15 × 1%, so the answer is 15 × 2 or 30. In the case of the 44 out of 200 who caught fish, we might reason that 44 = 22 × 2 (and 2 is 1% of the 200). So 44 is 22% of 200.

From the information in item 6, the total number of boys in the second camp can be found.

40% means $\frac{40}{100}$ of the group, and also refers to 32 boys.

We wish to find x such that $\frac{40}{100} = \frac{32}{x}$

Some of you can determine x by inspection, thus

$$\frac{40}{100} = \frac{32}{x}$$

$$\frac{40}{100} = \frac{32}{80}$$ So there are 80 boys in the second camp.

Another way to solve the problem is

$$\frac{40}{100} = \frac{32}{x}$$

$$40 = \frac{32}{x} \cdot 100$$

$$40x = 3200$$

$$x = \frac{3200}{40}$$

$$x = 80$$

A third way to solve the problem is to compare 40% with 10%. If 40% of the group is 32, then 10% would be $\frac{1}{4}$ of 32 or 8. If 10% of the group is 8, then 100% would be 10 × 8 or 80. In other words, we wish to find x such that

$$\frac{10}{100} = \frac{8}{x}$$

$$\frac{10}{100} = \frac{8}{80}$$

Exercises 8-2b

1. A Little League team won 3 out of the first 5 games played.
   (a) What percent of the first 5 games did the team win?
   (b) What percent of the first 5 games did the team lose?
   (c) What percent of the games played did the team win or lose?
2. Later in the season the team had won 8 out of 16 games played.
   (a) What was the percent of games won at this time?
   (b) Has the percent of games won increased or decreased?
3. At the end of the season, the team had won 26 games out of 40.
   (a) What percent of the games played did the team win by the end of the season?
   (b) How does this percent compare with the other two?

Exercises 4-10 all refer to the same junior high school.

4. There are 600 seventh grade pupils in a junior high school. The Principal hopes to divide the pupils into 20 sections of equal size.
   (a) How many pupils would be in each section?
   (b) What percent of the pupils would be in each section?
   (c) How many pupils is 1% of the number of pupils in the seventh grade?
   (d) How many pupils is 10% of the number of pupils in the seventh grade?

   Answers to (c) and (d) may help you with other answers.
5. Suppose one section contains 36 pupils. What percent of the seventh grade pupils are in that section?
6. One hundred fifty seventh grade pupils come to school on the school bus.

(a) What percent of the seventh grade pupils come by school bus?

(b) What percent of the seventh grade pupils come to school by some other means?

7. In a class of 30 pupils, 3 were tardy one day.

(a) What fraction of that class were tardy?

(b) What percent of that class were tardy?

8. In this junior high school there were 750 eighth grade pupils. The number of eighth grade pupils is what percent of the number of seventh grade pupils?

9. One day 3 seventh grade pupils came to school on crutches (they had been skiing.) What percent of the number of seventh grade pupils were on crutches?

10. One day a seventh grade pupil heard the Principal say "Four percent of the ninth graders are absent today." A list of absentees for that day had 22 names of ninth grade pupils on it. From these two pieces of information, the seventh grade pupil discovered the number of ninth grade pupils in the school. Can you find this number?

8-3. Decimal Notation

A numeral such as 3284 can be written as $3(10^3) + 2(10^2) + 8(10) + 4$. This is called expanded form. (See Chapter 2.) Written as 3284 the numeral is said to be written in positional notation. If the base is ten, as in the illustration above, this form is also called decimal notation. Each digit assumes a certain value according to its place in the numeral; that is, the 3 is in the thousands place, the 2 is in the hundreds place, and so on. The pattern for place value in base ten shows that each place immediately to the left of a given place is ten times the value of the given place (or each place immediately to the right of a given place is one-tenth of the value of the given place). Another way to look at this pattern is to recognize that reading from the left to the right, the exponents of 10 decrease.

Here is another example: $3(10^3) + 2(10^2) + 8(10^1) + 4 + 5(?) + 6(?) + 9(?)$. The 4 is in the units (or one) place. The value of this place is $\frac{1}{10}$ of the value of the place before it. If we extend our numeral to the right and still keep the pattern, what should be the value of the next place to the right? Of the next place after that one? The numeral in positional notation for this example is in two parts--3284 and 569. It is an example which illustrates expanded form to the right of the unit's place:

$$3(10^3) + 2(10^2) + 8(10^1) + 4 + 5\left(\frac{1}{10}\right) + 6\left(\frac{1}{10^2}\right) + 9\left(\frac{1}{10^3}\right)$$

The following chart shows the place values both to the left of and to the right of the units place.

Place Value Chart

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	Unit	Tenths	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10^1	10^0	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

When the above number expression is written in positional notation we might copy the digits in order and write each in its proper place. Thus we might write 3284569. However, this is not what we mean since there is no way to tell the whole number places from the fractional places. The punctuation mark used to locate the one's place is the decimal point. Thus the numeral is written 3284.569. This is read, "Three thousand two hundred eighty-four and five hundred sixty-nine thousandths" or "Three two eight four-point-five six nine."

Example 1. Write $5(10^2) + 7(10) + 3(\frac{1}{10})$ in decimal notation (that is, positional notation). 570.3.

Example 2. Write 42.306 in expanded form. $4(10) + 2(1) + 3(\frac{1}{10}) + 6(\frac{1}{10^3})$. Notice that the $(\frac{1}{10^2})$ place was not written. Why? Could it be included?

Exercises 8-3a

1. Write each of the following in decimal notation:
   (a) $6(10) + 5(1) + 8(\frac{1}{10}) + 7(\frac{1}{10^2})$
   (b) $4(10^2) + 3(10) + 6(1) + 1(\frac{1}{10}) + 9(\frac{1}{10^2})$
   (c) $5(10) + 2(\frac{1}{10}) + 4(\frac{1}{10^2})$
   (d) $4(\frac{1}{10}) + 8(\frac{1}{10^2}) + 3(\frac{1}{10^3})$
   (e) $2(\frac{1}{10^3}) + 6(\frac{1}{10^4})$
2. Write each of the following in expanded form:
   (a) 52.55
   (b) 1.213
   (c) 0.4

(d) 3.01

(e) 0.0102

3. Write each of the following in words:

   (a) 7.236

   (b) 0.004

   (c) 360.101

   (d) 1.0101

4. Write in decimal numeral form:

   (a) Three hundred and fifty-two hundredths

   (b) Five hundred-seven ten thousandths

   (c) Fourteen thousandths

   (d) Sixty and 7 hundredths

*5. Write 0.1_{two} in base ten.

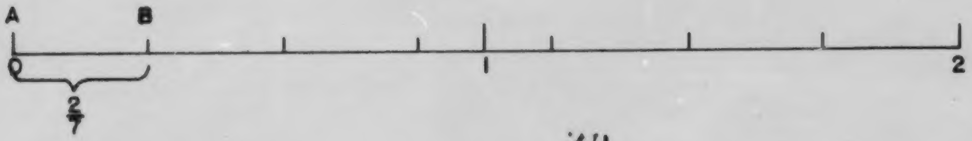
*6. Write $\frac{1}{2}$ in the duodecimal system.

*7. Write $\frac{5}{6}$ in the duodecimal system.

*8. Change 10.011_{two} to base ten.

In Chapter 6, a rational number was defined as the quotient of a whole number by a counting number, and this quotient was interpreted on the number line.

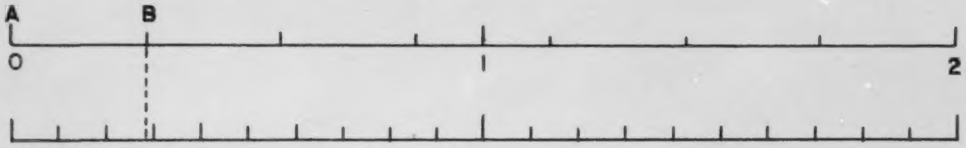
For example, the rational number $\frac{2}{7}$ was interpreted as follows: Divide a line segment 2 units long into 7 equal parts. The measure of the length of one of the parts is $\frac{2}{7}$. The procedure is illustrated below.



Suppose we want to express $\frac{2}{7}$ in decimal form. We know that $\frac{2}{7}$ is less than 1, so each of the digits to the left of the decimal point is zero. We need to find a digit for each place to the right of the decimal point. That is, we need to find the digits indicated by question marks in the following number sentence:

$$\frac{2}{7} = ?\left(\frac{1}{10}\right) + ?\left(\frac{1}{10^2}\right) + ?\left(\frac{1}{10^3}\right) + \ldots$$

Let us take them in order. First, we will find the digit in the tenths place. We want to find the length, in tenths, of the segment labeled $\overline{AB}$ in the figure.



Since a length of 2 units is 20 tenths of a unit, and we have divided the length of 2 units into 7 equal parts, each part must be $\frac{20}{7}$ tenths of a unit in length. But $\frac{20}{7} = 2\frac{6}{7}$, so the length of $\overline{AB}$, in tenths of a unit, is between 2 and 3. It is 2 tenths of a unit and some left over; the tenth's digit is "2".

Now let us find the hundredth's digit. The reasoning is easier if we start from the beginning. A length of 2 units is 200 hundredths of a unit and, since we have divided the length of 2 units into 7 equal parts, each part must be $\frac{200}{7}$ hundredths of a unit in length. But $\frac{200}{7} = 28\frac{4}{7}$, so that the length of $\overline{AB}$, in hundredths of a unit lies between 28 and 29. It is 2 tenths of a unit plus 8 hundredths of a unit and some left over; the hundredths's digit is "8".

Do you see the pattern? To find the thousandth's digit, notice that 2 units is 2000 thousandths of a unit, so the length of $\overline{AB}$ is $\frac{2000}{7}$ thousandths of a unit. But $\frac{2000}{7} = 285\frac{5}{7}$, so the length of $\overline{AB}$, in thousandths of a unit is between 285 and 286. It is 2 tenths of a unit plus 8 hundredths of a unit plus 5 thousandths of a unit and some left over; the thousandth's digit is "5".

The steps above can be done by division as follows:

$$\begin{array}{r} 0 \\ 7\,\overline{)2} \\ \underline{0} \\ 2 \end{array}$$

The whole number part of the quotient is zero. Each digit to the left of the decimal point is a zero.

$$\begin{array}{r} 0.2 \\ 7\,\overline{)2.0} \\ \underline{1\ 4} \\ 6 \end{array}$$

$\frac{2}{7}$ is 2 tenths and some left over.

$$\begin{array}{r} 0.28 \\ 7\,\overline{)2.00} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

$\frac{2}{7}$ is 28 hundredths and some left over.

$$\begin{array}{r} 0.285 \\ 7\,\overline{)2.000} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 5 \end{array}$$

$\frac{2}{7}$ is 285 thousandths and some left over.

Continuing the division:

$$\begin{array}{r} 0.285714 \\ 7\,\overline{)2.000000} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$

Enough steps have been shown to conclude that $\frac{2}{7}$ could never be written as an exact decimal.

Why?

You should do this type of problem by short division.

$$7\,\overline{)2.000000}\quad 0.285714$$

When rational numbers are changed to decimals it is often necessary to round the number.

Example 1. Find a decimal for $\frac{2}{7}$ to the nearest tenth. (Round to the nearest tenth.)

Look at 2 - 7 above.

In order for us to decide to the nearest tenth we look at the hundredth place. If the digit in the hundredth place is greater than 5 and the decimal has not terminated increase the numeral in the tenth place by one. Thus, $\frac{2}{7}$ rounded to the nearest tenth is 0.3.

If the question is being answered by performing the division, it is convenient to look at the remainder from the tenths place. If it is more than $\frac{1}{2}$ of the divisor, increase the last digit in the quotient by one. Otherwise, drop the remainder.

Example 2. Find a decimal for $\frac{2}{7}$ to the nearest ten-thousandth. (Round to the nearest ten-thousandth.)

Examine the digit in the hundred-thousandth place. In this case the digit is 1 and since this is less than 5, the preceding place is left as 7. Thus, $\frac{2}{7}$ to the nearest ten-thousandth is 0.2857.

If you were doing the division make this decision by looking at the remainder.

In rounding numbers, if necessary we always examine the remainder from the last subtraction in the division process or we look at the next digit in the decimal. If the digit is exactly 5 in a terminating decimal, a common agreement (especially among scientists) is: round to an even digit. Examples: Round 3.65 and 3.75 to tenths. The 3.65 becomes 3.6 and the 3.75 becomes 3.8. In each case we have rounded so that the last digit is even. Why would this rule be good for scientific calculations?

Exercises 8-3b

1. Write each of the following as a decimal.

   (a) $\frac{7}{4}$ (d) $\frac{15}{8}$

   (b) $\frac{8}{5}$ (e) $\frac{100}{16}$

   (c) $\frac{1}{8}$

2. Write each of the following as a decimal to the nearest hundredth place.

   (a) $\frac{1}{7}$ (d) $5\frac{1}{9}$

   (b) $\frac{2}{3}$ (e) $\frac{16}{3}$

   (c) $\frac{5}{9}$

3. Round each of the following to the nearest tenth place.

   (a) 16.38 (d) 0.034

   (b) 48.72 (e) 0.051

   (c) 108.05 (f) 1.16

4. Round each of the following to the nearest thousandth.

   (a) 4.0486 (c) 0.0006

   (b) 17.1074 (d) 0.00049

5. (a) Round $\frac{3}{7}$ to the nearest tenth; hundredth; thousandth.
   (b) Make a sketch on a number line of each of the rounded numbers.
   (c) Which is closest to the actual number $\frac{3}{7}$?
   (d) Which is least like the actual number $\frac{3}{7}$?

6. Change each number of the pair to a decimal and rank them, place the larger first.
   (a) $\frac{3}{7}$; $\frac{17}{42}$
   (b) $\frac{5}{16}$; $\frac{26}{83}$

*7. When we discussed writing $\frac{2}{7}$ as a decimal it was compared to the problem of naming a point on a line. The explanation of changing a fraction like $\frac{2}{7}$ to a decimal could have been made by using expanded notation form. The problem would be: $?(10) + ?(1) + ?(\frac{1}{10}) + ?(\frac{1}{10^2}) + ?(\frac{1}{10^3})$ ... Since each place to the right is one tenth of the preceding place we always asked the question: "How many one-tenths are there in the small amount that was left over from the previous calculation?" We were finding numerals for the question marks. Use this idea to answer the following:
   (a) Write $\frac{3}{2}$ in base two using a (.).
   (b) Write $\frac{1}{4}$ in base two using a (.).
   (c) Write $\frac{1}{3}$ in base two using a (.).
   (d) Write $\frac{1}{3}$ in the duodecimal system.
   (e) From (c) and (d) does the fact that a decimal "comes out exact" or not depend upon the number or the base in which the number is written or both?

(f) Give one base other than 12 which would make $\frac{1}{3}$ come out exact.

*8. Find the sum of the following series: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$

8-4. Arithmetic Operations with Decimals

Suppose we wish to add two decimals, for example, 0.73 + 0.84. We know how to add these numbers without the decimal places: 73 + 84 = 157. How do we handle the decimal places? We may proceed as follows.

$$0.73 = 73 \times \frac{1}{100} \text{ and } 0.84 = 84 \times \frac{1}{100}.$$

Therefore,

$$\begin{aligned} 0.73 + 0.84 &= (73 \times \frac{1}{100}) + (84 \times \frac{1}{100}) \\ &= (73 + 84) \times \frac{1}{100} \\ &= 157 \times \frac{1}{100} = 1.57. \end{aligned}$$

Notice that we have used the distributive property here.

Suppose now we wish to add a two decimal place number and a three decimal place number, for example, 0.73 + 0.125. Just as before we first rewrite our numbers as fractions:

$$0.73 = 73 \times \frac{1}{100} = 730 \times \frac{1}{1000} \text{ and } 0.125 = 125 \times \frac{1}{1000}.$$

We use the last form of .73, since then the factor $\frac{1}{1000}$ will appear in both numbers.

$$\begin{aligned} 0.73 + 0.125 &= (730 \times \frac{1}{1000}) + (125 \times \frac{1}{1000}) \\ &= (730 + 125) \times \frac{1}{1000} \\ &= 855 \times \frac{1}{1000} = 0.855. \end{aligned}$$

These examples can be handled more conveniently by writing one number below the other as follows.

$$\begin{array}{r} .73 \\ +\ .84 \\ \hline 1.57 \end{array} \qquad\qquad \begin{array}{r} 0.73 \\ +\ 0.125 \\ \hline 0.855 \end{array}$$

Notice that we write the decimal points directly under one another This is because we want to add the digit in the $\frac{1}{10}$ place in the first number to the digit in the $\frac{1}{10}$ place in the second, and the digit in the $\frac{1}{100}$ place in the first number to the digit in the $\frac{1}{100}$ place in the second, etc. Thus,

$$0.73 = \frac{7}{10} + \frac{3}{100} \quad \text{and} \quad 0.84 = \frac{8}{10} + \frac{4}{100}$$

and therefore

$$\begin{aligned} 0.73 + 0.84 &= \frac{7+8}{10} + \frac{3+4}{100} \\ &= \frac{15}{10} + \frac{7}{100} \\ &= 1.57 \end{aligned}$$

Subtraction can be handled in the same way. For example,

$$\begin{array}{r} 0.84 \\ -\ 0.73 \\ \hline 0.11 \end{array} \qquad\qquad \begin{array}{r} 0.83 \\ -\ 0.74 \\ \hline .09 \end{array}$$

Exercises 8-4a

1. Add the following numbers.
   (a) 0.76 + 0.84
   (b) 0.719 + 0.382
   (c) 1.002 + 0.00102
   (d) 1.05 + 0.75 + 21.5

2. Subtract the following numbers.

   (a) 0.84 - 0.76

   (b) 0.625 - 0.550

   (c) 0.500 - 0.125

   (d) 1.005 - 0.0005

3. Perform the operations indicated.

   (a) 1.051 - 0.702 + 0.066

   (b) 0.4075 - 0.32 + 0.076

4. Four men enter a hardware store, and the first wants to buy 10.1 feet of copper wire, the second wants 15.1 feet, the third wants 8.6 feet, and the fourth wants 16.6 feet. The storekeeper has 50 feet of wire in his store. Can he give each man what he wants?

5. A storekeeper has 11.5 pounds of sugar. A woman buys 5.6 pounds. Another woman buys 4.8 pounds. Then a delivery truck brings 25 pounds to the store. Finally, mice eat 0.05 pounds. How much sugar is left?

6. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 lb?

*7. Add $10.01_{two} + 1.01_{two}$ and then express the answer in the base ten.

   Note. In the base two: $0.1_{two} = \frac{1}{2}$ and $0.01_{two} = \frac{1}{4}$.

Suppose we wish to multiply two decimals. For example, 0.3 × 0.25. We know how to multiply these numbers without the decimal places: 3 × 25 = 75. Just as before we write

$$0.3 \times 0.25 = 3 \times \frac{1}{10} \times 25 \times \frac{1}{100}$$

$$= 3 \times 25 \times \frac{1}{10} \times \frac{1}{100}$$

$$= 75 \times \frac{1}{1000}$$

$$= 0.075$$

(1) How many digits are there to the right of the decimal point in 0.3?

(2) How many digits are there to the right of the decimal point in 0.25?

(3) What is the sum of the answers to (1) and (2)?

(4) How many digits are there to the right of the decimal point in 0.075?

(5) Compare the answers to (3) and (4).

Now multiply 0.4 × 0.25. What is your answer? Answer the five questions above, (1), (2), (3), (4), (5) for these numbers. Do the answers to (3) and (4) still agree?

Property 2: To find the number of decimal places when two numbers are multiplied, add the number of decimal places in the two numbers.

For example, suppose we wish to multiply .732 by .25. The first number has three decimal places and the second has two, so there will be five decimal places in the answer. We multiply out 732 × 25, and then mark off five decimal places in the answer, counting from right to left.

$$\begin{array}{r} .732 \\ \underline{.25} \\ 3660 \\ \underline{1464} \\ .18300 \end{array}$$

We consider now the problem of dividing one decimal number by another, for example, .125 ÷ .5.

$$
\begin{aligned}
\frac{.125}{.5} &= \frac{\frac{125}{1000}}{\frac{5}{10}} \\
&= \frac{125}{1000} \times \frac{10}{5} \\
&= \frac{125}{5} \times \frac{10}{1000} \\
&= 25 \times \frac{1}{100} \\
&= .25
\end{aligned}
$$

Let us try another example, $.5 \div .04$. From Example 1 in Section 8-3 we know that $5 \div 4 = 1.25$. Therefore we have

$$
\begin{aligned}
\frac{.5}{.04} &= \frac{\frac{5}{10}}{\frac{4}{100}} \\
&= \frac{5}{10} \times \frac{100}{4} \\
&= \frac{5}{4} \times \frac{100}{10} \\
&= 1.25 \times 10 \\
&= 12.5
\end{aligned}
$$

Finally, let us try an example with a repeating decimal. From Example 2 of Section 8-3 we know that $2 \div 7 = 0.285714285714\ldots$ where 285714 is repeated over and over again, without end.

Let us try the problem: $.02 \div .7$.

$$
\begin{aligned}
\frac{.02}{.7} &= \frac{\frac{2}{100}}{\frac{7}{10}} \\
&= \frac{2}{100} \times \frac{10}{7} \\
&= \frac{2}{7} \times \frac{10}{100} \\
&= .285714\ldots \times \frac{1}{10} \\
&= .0285714
\end{aligned}
$$

In Chapter 6 we had the following property

> If the numerator and denominator of a fraction are both multiplied by the same number, the number which the fraction represents is unchanged.

Thus $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$. We shall use this property for fractions with decimals. For example,

$$\frac{.125}{.5} = \frac{1.25}{5} \quad \text{and} \quad \frac{.02}{.7} = \frac{.2}{7}.$$

Here we have multiplied the numerator and the denominator by 10. Another example is

$$\frac{.5}{.04} = \frac{50}{4}.$$

By what number were numerator and denominator multiplied in this example?

In dividing one decimal by another we first shift all digits in the divisor to the left of the decimal point by multiplying dividend and divisor by a suitable power of ten. Thus:

$$.125 \div .5 = 1.25 \div 5, \quad .5 \div .04 = 50 \div 4, \text{ and}$$
$$.02 \div .7 = .2 \div 7.$$

After all digits in the divisor have been shifted to the left of the decimal point the division is performed.

$$\begin{array}{r} .25 \\ 5\,\overline{)1.25} \\ \underline{1\ 0} \\ 25 \\ \underline{25} \\ 0 \end{array} \qquad \begin{array}{r} 12.5 \\ 4\,\overline{)50.0} \\ \underline{4} \\ 10 \\ \underline{8} \\ 20 \\ \underline{20} \\ 0 \end{array} \qquad \begin{array}{r} .0285 \\ 7\,\overline{).2000} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 5 \end{array}$$

These three examples should all be done by short division:

$$\begin{array}{r} .25 \\ 5\,\overline{)1.25} \end{array} \qquad \begin{array}{r} 12.5 \\ 4\,\overline{)50.0} \end{array} \qquad \begin{array}{r} .0285 \\ 7\,\overline{).2000} \end{array}$$

Notice that the decimal point in the answer is directly above the decimal point in the dividend.

Exercises 8-4b

(Assume that the decimals are exact.)

1. Find the following products.

   (a) $.009 \times .09$

   (b) $.0025 \times 2.5$

   (c) 1.2×120

   (d) $.135 \times .202$

2. Find the following quotients.

   (a) $.009 \div 30$

   (b) $.015 \div .05$

   (c) $.575 \div .4$

   (d) $2.04 \div .008$

3. Express the following fractions as decimals.

   (a) $\frac{3000}{8}$ (b) $\frac{300}{8}$ (c) $\frac{30}{8}$ (d) $\frac{3}{8}$ (e) $\frac{3}{80}$ (f) $\frac{3}{800}$

4. $\dfrac{.015 \times .0025 \times 2.5}{.05 \times .03} = ?$

5. John's father has a garden 25.3 feet long and 15.7 feet wide. How many square feet are there in the garden?

*6. Find the following product of two base seven numbers:

$$1.14_{\text{seven}} \times 2.4_{\text{seven}}.$$

Remember, $1.14_{\text{seven}} = 1 + \frac{1}{7} + \frac{4}{7^2}$ and $2.4 = 2 + \frac{4}{7}$.

Sometimes in dealing with decimal numbers we do not need all of the decimal places. For example, some machines will cut metal rods with length accurate to two decimal places but no more. Thus you could take a long rod and cut off 3.45 inches. But if you tried to cut off exactly 3.452 inches you could not be sure of getting it right. Even a good cutter will slip a little and make a small error.

If we are only interested in two decimal places and we have a number with more than two places, we "round it" to the nearest two places. For example, .238 is closer to .24 than to .23, and so we round it off to .24. In general, if the third place is more than 5 we agree to add one to the second place when we round it. If the third place is less than 5, we do not change the second place when we round. Here are some further examples.

Number	Rounded to two places
.234	.23
.237	.24
.241	.24
.244	.24
.1949	.19

If the third digit is 5 we agree to round in such a way that the second digit of the rounded number is even.

Thus .235 is rounded to .24, and .245 is also rounded to .24. The reason for this is explained after the following examples.

Number	Rounded to two places
.325	.32
.335	.34
.345	.34
.355	.36

When we round a number we deliberately make an error.

Thus when we round .325 to get .32 our answer is smaller than the original number. When we round .335 to get .34 our answer is bigger than the original number. This error may be increased when we add or multiply numbers. For example,

$$.16 + .16 = .32.$$

However, if we had first rounded these numbers to one place we would have

$$.2 + .2 = .4.$$

The error will sometimes cancel out. Thus

$$.245 + .245 = .500$$

If we had first rounded to two places we would have

$$.24 + .26 = .50.$$

The rule about rounding when the third digit is 5 was made to increase the chances of the errors cancelling out. If you are rounding a lot of numbers, then about half the time the rounded number will be bigger than the original number, and half the time it will be smaller. Thus when we add or multiply rounded numbers some of the errors will cancel out.

Sometimes a number is rounded to just one decimal place. Thus

$$.15, \quad .21, \quad .24$$

all round to .2.

We may also round to three places. Thus .2345 to .234, while .2355 rounds to .236.

If we want to get two decimal places correctly in an answer, we compute three places and then round. Thus $\frac{2}{7} = 0.285714\ldots$

If we round it to one place we get .3

Correct to two places we have .29

Correct to three places we have .286

Correct to four places we have .2857.

Exercises 8-4b

1. Round the following numbers to two places.
   - (a) .0351
   - (b) .0449
   - (c) .0051
   - (d) .0193
2. Round the following numbers to three places.
   - (a) 0.1599
   - (b) .0009
   - (c) .00009
   - (d) .3249
3. Express the following fractions as decimals correct to three places.
   - (a) $\frac{3}{8}$
   - (b) $\frac{1}{4}$
   - (c) $\frac{2}{3}$
4. Express the following fractions as decimals correct to one place.
   - (a) $\frac{7}{23}$
   - (b) $\frac{6}{23}$
   - (c) $\frac{2}{23}$
   - (d) $\frac{1}{23}$

5. (a) A piece of land is measured and the measurements are rounded to the nearest tenth of a foot (in other words, to the first place). The length, after rounding is 11.1 rods and the width is 3.9 rods. Compute the area.

(b) Suppose that the correct length is 11.14, and the correct width is 3.94 feet. Compute the correct area. What is the difference between this answer and the previous one?

8-5. Percent

You have learned that 50% is another name for the number $\frac{1}{2}$, because 50 $= \frac{50}{100} = \frac{1}{2}$. Also, we can write $\frac{50}{100}$ as the decimal .50. Sometimes it is convenient to use each of these four numerals. You have changed fractions to hundredths and then to percent. Suppose that you wish to change a percent such as 65% to a decimal or to a fraction in simplest form.

$$65\% = \frac{65}{100} = .65 = \frac{13}{20}$$

$$72\% = \ ? = \ ? = \ ?$$

Suppose that you wish to change a decimal such as .28 to a percent or to a fraction in simplest form

$$.28 = \frac{28}{100} = 28\% = \frac{7}{25}$$

$$.15 = \ ? = \ ? = \ ?$$

If one of the four names of a number is shown, you can determine the other three names.

$\frac{1}{4} = \frac{25}{100} = .25 = 25\%$. These are all names for one number.

$$\frac{1}{5} = \ ? = \ ? = \ ?$$

? = ? = ? = 10

? = ? = .75 = ?

In some cases the first two numerals as shown above are the same. For instance $\frac{3}{100}$, or .03, 3% are the only number names that will fit in the table for the number three hundredths. So $\frac{3}{100}$ belongs in each of the first two positions.

Exercises 8-5a

1. Fill in the missing names of numbers in the chart below. The completed chart will be helpful to you in future lessons.

Fraction Simplest form	Hundred as denominator	Decimal	Percent
(a) $\frac{1}{2}$	$\frac{50}{100}$	.50	50%
(b) $\frac{1}{4}$			
(c)	$\frac{75}{100}$		
(d)		.20	
(e)			40%
(f)	$\frac{60}{100}$		
(g) $\frac{4}{5}$			
(h)		.33...	
(i)	$\frac{70}{100}$		
(j)		.66...	
(k) $\frac{3}{10}$			
(l)		.10	
(m)			90%
(n) $\frac{1}{8}$			

Fraction Simplest form	Hundred as denominator	Decimal	Percent
(o)	$\frac{300}{100}$		
(p)		.375	
(q)			150%
(r)	$\frac{62.5}{100}$		
(s)		.01	
(t) $\frac{7}{8}$			
(u)			100%
(v)	$\frac{16\frac{2}{3}}{100}$		
(w) $\frac{5}{6}$			

2. Draw a number line and locate the percents in problem 1 on it.

3. Using squared paper, draw a large square containing 100 small squares. By proper shading locate the percents in parts (b), (d), (l), (p), (s).

4. What fraction in simplest form is another name for

(a) 32% (b) 90% (c) 120%

5. Give the percent names for the following numbers

(a) $\frac{13}{25}$ (b) $\frac{7}{20}$ (c) $\frac{19}{20}$ (d) $\frac{3}{10}$

Percent is used to express ratios of numerical quantities in everyday experience. It is important for you to understand the notation of percent, and, also, to be accurate in the computational use of percent.

Suppose that a family has an annual income of \$4860 (after withholding tax). The family budget includes an item for food of 32%. How much money is allowed for food for the year? This means that we wish to find x such that

$$\frac{x}{4860} = \frac{32}{100} \qquad \text{or} \qquad \frac{x}{4860} = .32$$

Before solving the equation we should estimate an approximate answer using some information about percents and fractions from the chart in Exercise 8-5a. Notice that 32% is a bit smaller than $33\frac{1}{3}\%$ (another name for $\frac{1}{3}$). One third of \$4500 is \$1500. So our answer should be close to \$1500. We will show two ways of writing the steps of solution. Property 9 from Chapter 6 is used in the solution.

$$\begin{aligned} \frac{x}{4860} &= \frac{32}{100} & \qquad \frac{x}{4860} &= .32 \\ x &= \frac{32}{100}\cdot 4860 & x &= 4860\cdot .32 \\ x &= 1555.20 \quad (\$1555.20) & x &= 1555.20 \end{aligned}$$

This answer seems reasonable when compared to our estimate of \$1500.

Suppose that this same family rents a house for \$77.00 per month. What percent of the family income will be spent for rent? First, we find that the rent for the year is \$924.00. If x is the percent that 924 is of 4860, then we wish to find x such that $\frac{924}{4860} = \frac{x}{100}$. To estimate the answer, notice that 900 is $\frac{1}{5}$ or 20% of 4500. Two ways of writing the steps in the solution are shown.

$$\begin{aligned} \frac{924}{4860} &= \frac{x}{100} \\ x &= 100 \cdot \frac{924}{4860} \\ x &\approx 19.01 \end{aligned}$$

$$\begin{aligned} \frac{924}{4860} &= \frac{x}{100} \\ 4860\,x &= 92400 \\ x &\approx 19.01 \end{aligned}$$

In this problem the answer will be rounded to 19%. Is this reasonable?

An advertisement said that a bicycle could be purchased "on time" by making a down payment of $14.70. The merchant stated further, that this payment was 26% of the price. What was the price of the bicycle? If the price of the bicycle is x dollars, we wish to find x such that

$$\begin{aligned} \frac{14.70}{x} &= \frac{26}{100} \\ 26x &= 1470 \\ x &\approx 56.538\ldots \end{aligned}$$

($56.54 would be the price. Why?)

How might you have estimated the answer? The answer $56.54 rounded to the nearest 10 cents would be $56.50. Which price do you think would be shown on the price tag?

Commissions and Discounts

People who work as salesmen often are paid a commission instead of a salary. A book salesman is paid a commission of 25% of the selling price of the set of books. If he sells a set of books for $60.00, his commission would be 25% of $60.00 or $15.00. How did we find this answer? Sometimes the percent of the selling price which gives the commission is called the rate of the commission.

Definition. Commission is the payment, often based on a percent of selling price, that is paid to a salesman for his services.

Merchants sometimes sell articles at a discount. During a sale, an advertisement stated "All coats will be sold at a discount of 30%." A coat marked \$70.00 then has a discount of 30% of \$70.00 or \$21.00. The sale price (sometimes called the net price) is \$70.00 - \$21.00 or \$49.00.

Definition. Discount is the amount subtracted form the marked price.

Definition. Sale price or net price is the marked price less the discount.

Exercises 8-5b

In the following problems it may be necessary to round some answers. Round money answers to the nearest cent, and round percent answers to the nearest whole percent.

1. On an examination there was a total of 40 problems. The teacher considered all of the problems of equal value, and assigned grades by percent. How many correct answers are indicated by the following grades?

   (a) 100% (b) 80% (c) 50% (d) 65%

2. In grading the papers what percent would the teacher assign for the following papers?

   (a) All problems worked, but 10 answers are wrong.
   (b) 36 problems worked, all answers correct.
   (c) 20 problems worked, 2 wrong answers.
   (d) One problem not answered, and 1 wrong answer.

3. If the sales tax in a certain state is 4% of the purchase price, what tax would be collected on the following purchases?

(a) A dress selling for $17.50

(b) A bicycle selling for $49.50

4. A real estate agent receives a commission of 5% for any sale that he makes. What would be his commission on the sale of a house for $17,500?

5. This real estate agent wishes to earn an annual income from commissions of at least $9000. To earn this income, what would his yearly sales need to total?

6. A salesman who sells vacuum cleaners earns a commission of $25.50 on each one sold. If the selling price of the cleaner is $85.00 what is the rate of commission for the salesman?

7. Sometimes the rate of commission is very small. Salesmen for heavy machinery often receive a commission of 1%. If, in one year, such a salesman sells a machine to an industrial plant for $658,000 and another machine for $482,000, has he earned a good income for the year?

8. A sports store advertised a sale of football equipment. The discount was to be 27%.

   (a) What would be the sale price of a football whose marked price is $5.98?

   (b) What would be the sale price of a helmet whose marked price is $3.40?

9. In a junior high school there are 380 seventh grade pupils, 385 eighth grade pupils, and 352 ninth grade pupils.

   (a) What is the total enrollment of the school?

   (b) What percent of the enrollment is in the seventh grade?

(c) What percent of the enrollment is in the eighth grade?

(d) What percent of the enrollment is in the ninth grade?

What is the sum of the answers to (b), (c), and (d)?

10. Mr. Martin keeps a record of the amounts of money his family pays in sales tax. At the end of one year he found that the total was \$96.00 for the year. If the sales tax rate is 4%, what was the total amount of taxable purchases made by the Martin family during the year?

Percents used for comparison

In problem 1 of Exercises 8-5a, some of the percents were not whole percents. The fraction $\frac{1}{8}$ changed to percent is $12\frac{1}{2}\%$ You have learned that $\frac{1}{2}\%$ can be read $\frac{1}{2}$ of 1%. Decimal percents are often used, such as .7%.

$$.7\% (.7 \text{ of } 1\%) = .7 \times \frac{1}{100} = \frac{.7}{100} = \frac{7}{1000} = .007$$

These are all names for the same number. If we wish to find .7% of \$300, we wish to find x such that

$$\frac{x}{300} = \frac{.7}{100} \qquad \text{or} \qquad \frac{x}{300} = .007$$

$$100x = 210 \qquad\qquad x = 300 \cdot .007$$

$$x = 2.10 \ (\$2.10) \qquad\qquad x = 2.10 \ (\$2.10)$$

.7% is less than 1%. Since 1% of \$300 is \$3.00, the answer \$2.10 is reasonable.

Suppose that we wish to find 2.3% of \$500.

$2.3\% = \frac{2.3}{100} = \frac{23}{1000} = .023$. Find the number such that

$$\frac{x}{500} = \frac{2.3}{100} \qquad \text{or} \qquad \frac{x}{500} = .023$$

$$100x = 1150 \qquad\qquad x = 500 \cdot .023$$

$$x = 11.50 \ (\$11.50) \qquad\qquad x = 11.50 \ (\$11.50)$$

Baseball batting averages are found for each player by dividing the number of hits he makes by the number of times he has been at bat. The division is carried to the nearest thousandth. So the batting average can be considered as a percent expressed to the nearest tenth of a percent. If a player has 23 hits out of 71 times at bat, his batting average is $\frac{23}{71}$ or .324.

Sometimes answers are called for to the nearest tenth of a percent. A teacher might be asked what percent of his grades are B. The teacher issues 163 grades, 35 of them B. He wishes to find x such that $\frac{35}{163} = \frac{x}{100}$.

$$\begin{aligned} \frac{35}{163} &= \frac{x}{100} \\ x &= \frac{35}{163} \cdot 100 \\ x &= 21.47\ldots \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{35}{163} &= \frac{x}{100} \\ 163x &= 3500 \\ x &= 21.47\ldots \end{aligned}$$

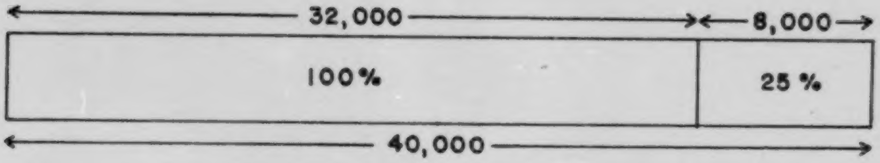
In section 8-3 you learned how to round decimals. If the percent is called for to the nearest tenth of a percent, 21.47... would round to 21.5%.

Percents of increase and decrease

Percent is used to indicate an increase or a decrease in some quantity. Suppose that Central City had a population of 32,000 (rounded to the nearest thousand) in 1950. If the population increases to 40,000 by 1960, what will be the percent of increase?

$$\begin{array}{r} 40{,}000 \\ -\ \underline{32{,}000} \\ 8{,}000 \end{array} \text{ (actual increase)} \qquad \begin{aligned} \frac{8{,}000}{32{,}000} &= \frac{x}{100} \\ x &= \frac{8{,}000}{32{,}000} \cdot 100 \\ x &= 25 \text{ (increase of 25\%)} \end{aligned}$$

Notice that the percent of increase is computed by comparing the actual increase to the earlier population figure.



The 40,000 is made up of the 32,000 (100%) plus the increase of 8,000 (25%). So the population of 40,000 in 1960 is 125% of the population of 32,000 in 1950.

Suppose that Hill City had a population of 15,000 in 1950. If the population in 1960 is 12,000, what will be the percent of decrease? If x represents the percent of decrease, then

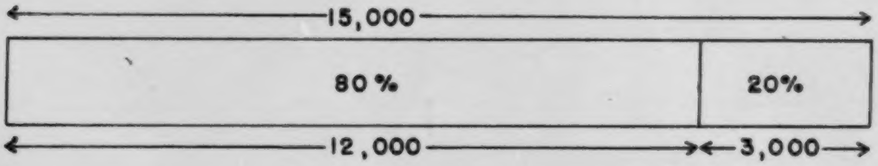
$$\begin{array}{r} 15{,}000 \\ -\ \underline{12{,}000} \\ 3{,}000 \end{array} \text{ actual decrease}$$

$$\frac{3{,}000}{15{,}000} = \frac{x}{100}$$

$$x = 100 \cdot \frac{3{,}000}{15{,}000}$$

$$x = 20 \text{ (decrease of 20\%)}$$

Notice that the population decrease also is computed by comparing the actual decrease to the earlier population figure.



The 12,000 is the difference between 15,000 (100%) and the decrease of 3,000 (20%). So the population in 1960 of 12,000 is 80% of the population of 15,000 in 1950.

If the rents in an apartment house are increased 5%, each tenant can compute his new rent. Suppose that a tenant is paying $80 for rent, what will he pay in rent after the increase? If x represents the increase in rent, then

$$\frac{x}{80} = \frac{5}{100}$$

$$x = 80 \cdot \frac{5}{100}$$

$$x = 4 \quad \text{(increase of \$4.00)}$$

The new rent will be \$80 + \$4 = \$84.

Exercises 8-5c

1. A junior high school mathematics teacher had 176 pupils in his 5 classes. The semester grades of the teacher were A 20; B 37; C 65; D 40; E 14.
   (a) What percent of the grades were A? (to nearest tenth percent)
   (b) What percent of the grades were B?
   (c) What percent of the grades were C?
   (d) What percent of the grades were D?
   (e) What percent of the grades were E?
   (f) What is the sum of the answers in parts (a), (b), (c), (d), (e)? Does this sum help check the answers?
2. Bob's weight increased during the school year from 65 pounds to 78 pounds. What was the percent of increase?
3. During the same year, Bob's mother reduced her weight from 160 pounds to 144 pounds. What was the percent of decrease?
4. The enrollment in a junior high school was 1240 in 1954. In 1959 the enrollment had increased 25%. What was the enrollment in 1959?
5. Jean earned \$14.00 during August. In September she earned only \$9.50. What was the percent of decrease in her earnings?

6. A salesman of heavy machinery earned a commission of $4850 on the sale of a machine for $970,000.
   - (a) At what rate is his commission paid?
   - (b) What will be his commission for the sale of another machine for $847,500?

7. James was 5 ft. tall in September. In June his height is 5 ft. 5 in. Both heights were measured to the nearest inch. What is the percent of increase in height?

8. Do you know your height at the beginning of the school year? Now? Do you know your weight at the beginning of the school year? Now?
   - (a) What is the percent of increase in your height since last September?
   - (b) What is the percent of increase in your weight since last September?

9. A baseball player named Jones made 25 hits out of 83 times at bat. Another player named Smith made 42 hits out of 143 times at bat.
   - (a) What is the batting average of each player?
   - (b) Which player has the better record?

10. An elementary school had an enrollment of 790 pupils in September, 1955. In September, 1959, the enrollment was 1012. What was the percent of increase in enrollment?

Two methods for solving problems of percent of increase and decrease

In the solution of problems involving percent of increase or percent of decrease, two approaches can be used.

If the cost of butter increases from 80¢ a pound to 92¢ a pound, what is the percent of increase? The method you have been using is

92¢ - 80¢ = 12¢ (increase). If 12 is x percent of 80, then

$$\frac{12}{80} = \frac{x}{100}$$

$$x = 100 \cdot \frac{12}{80}$$

$$x = 15 \text{ or } 15\% \text{ increase}$$

A second method finds what percent 92¢ is of 80¢. Since 92 is larger than 80, then 92 is more than 100% of 80. If 92 is x percent of 80, then

$$\frac{92}{80} = \frac{x}{100}$$

$$x = 100 \cdot \frac{92}{80}$$

$$x = 115 \text{ or } 115\%$$

This means that 92¢ is 115% of 80¢. If 100% is subtracted from 115%, the percent of increase (15%) results.

In a certain city the fire department extinguished 160 fires during 1958. During 1959, the number of fires extinguished dropped to 120. What was the percent of decrease? We will show two ways to solve this problem. In one method we find what percent the difference (160 - 120) is of 160. In the other method we find what percent the number of fires in the later year is of the number of fires in the earlier year. This percent is then compared with 100%. If 120 is y percent of 160, then

$$\begin{array}{r} 160 \\ -\ \underline{120} \\ 40 \end{array}$$ (fewer fires)

If 40 is x percent of 160, then

$$\frac{40}{160} = \frac{x}{100}$$

$$x = 100 \cdot \frac{40}{160}$$

$$x = 25 \text{ or } 25\%$$

There was a decrease of 25% in the number of fires in 1959 when compared with the number in 1958.

$$\frac{120}{160} = \frac{y}{100}$$

$$y = 100 \cdot \frac{120}{160}$$

$$y = 75 \text{ or } 75$$

$$100\% - 75\% = 25\%$$

The later number of fires was 75% of the earlier number. There was a decrease of 25% for the number of fires in 1959 when compared with the number in 1958.

Of course, the answers should be the same for the two methods of solution.

Exercises 8-5d

In each problem, 1 through 5 compute the percent of increase or decrease by both methods. If necessary, round percents to the nearest tenth of a percent.

1. In a junior high school the lists of seventh grade absentees for a week numbered 29, 31, 32, 28, 30. The next week the five lists numbered 22, 26, 24, 25, 23.
   (a) What was the total number of pupil days of absence for the first week?

(b) What was the total for the second week?

(c) Compute the percent of increase or decrease in the number of pupil days of absence.

2. On the first day of school a junior high school had an enrollment of 1050 pupils. One month later the enrollment was 1200. What was the percent of increase?

3. One week the school lunchroom took in $450. The following week the amount was $425. What was the percent of decrease?

4. From the weight at birth, a baby's weight usually increases 100% in six months.

   (a) What should a baby weigh at six months, if its weight at birth is 7 lb. 9 oz.?

   (b) Suppose that the baby in (a) weighs 17 lb. at the age of six months. What is the percent of increase?

5. During 1958 a family spent $1490 on food. In 1959 the same family spent $1950 on food. What was the percent of increase in the money spent for food?

*6. During 1958 the owner of a business found that sales were below normal. The owner announced to his employees that all wages for 1959 would be cut 20%. By the end of 1959 the owner noted that sales had returned to the 1957 levels. The owner then announced to the employees that the 1960 wages would be increased 20% over those of 1959.

   (a) Which of the following statements is true?

      (1) The 1960 wages are the same as the 1958 wages.

      (2) The 1960 wages are less than the 1958 wages.

      (3) The 1960 wages are more than the 1958 wages.

(b) If your answer to part (a) is (1), justify your answer. If your answer to part (a) is (2) or (3) express the 1960 wages as a percent of 1958 wages.

*7. In an automobile factory the number of cars coming off the assembly line in one day is supposed to be 500. One week the plant operated normally on Monday. On Tuesday there was a breakdown which decreased the number of completed cars to 425 for the day. On Wednesday operations were back to normal.

(a) What was the percent of decrease in production on Tuesday compared with Monday?

(b) What was the percent of increase in production on Wednesday compared to Tuesday?

8-6. Decimal Expansion

We have learned how to name a rational number by a decimal numeral (or decimal). Two questions which present themselves now are these:

"May every rational number be named by a decimal numeral? Does every decimal numeral name some rational number?"

Let's think about the first question. We may begin by choosing some fractions which name rational numbers and see if we can find a decimal numeral which names the same rational number.

$\frac{1}{8}$ is the quotient of 1 divided by 8.

$$8\,\overline{)\,1.000}^{\;0.125}$$

So $\frac{1}{8}$ and 0.125 are names of the same rational number.

We say that a decimal numeral ends or terminates if it is obtained by a division which is exact or in which there occurs a zero remainder at some stage in the division process. In this sense the decimal numeral 0.125 which names the rational number $\frac{1}{8}$ is a terminating decimal.

Consider the rational number $\frac{1}{3}$. We all know the decimal numeral which names $\frac{1}{3}$. It is obtained by dividing 1 by 3.

$$\begin{array}{r} 0.3333\ldots \\ 3\,\overline{)1.0000} \end{array}$$

Can you tell without performing the division, the digits that should appear in each of the next six places?

Does this decimal finally end (terminate)? That is, at some stage in the division do we obtain a zero remainder?

Let us use the dots to indicate that the decimal numeral never ends, but goes on and on without end.

Let's try to find a decimal numeral to name the rational number $\frac{1}{7}$.

$$\begin{array}{l} \;\;\;\;0.142857142857\ldots \\ 7\,\overline{)1.00000000000000}\ldots \end{array}$$

Class Discussion

1. Can you tell, without performing the division, the digits that should appear in each of the next six places?
2. Does this decimal numeral finally end? (That is, at some stage in the division do we obtain a zero remainder?)
3. Is there a block of digits which continues to repeat endlessly? Let us place a horizontal line over this block of digits which repeats. Thus $0.142857\overline{142857}\ldots$ The dots indicate that the block of digits never ends, but is repeated endlessly.

4. Name $\frac{1}{11}$ by a decimal numeral.
5. How soon can you recognize a pattern?
6. Does this decimal numeral end? How should you indicate this?
7. Is there a set of digits which repeats periodically? How should you indicate this?

Exercises 8-6a

1. Write a decimal numeral for $\frac{1}{13}$.
   - (a) How soon can you recognize a pattern?
   - (b) Does this decimal numeral end?
   - (c) How should you indicate that it does not end?
   - (d) Is there a set of digits which repeats periodically?
   - (e) How should you indicate this?
2. Write decimal numerals for:

   (a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$

See how soon you can recognize a pattern in each case. In performing the division watch the remainders. They may give a clue about when to expect the decimal numeral to begin to repeat.

Exercises 8-6b

1. Write decimal numerals for:

   (a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{9}{11}$ (e) $\frac{14}{11}$ (f) $\frac{23}{11}$
2. Study these decimal numerals and see if you can find a relationship
   - (a) between the decimal numeral naming $\frac{1}{11}$ and the decimal numeral naming $\frac{2}{11}$;

(b) between the decimal numeral naming $\frac{1}{11}$ and the decimal numeral naming $\frac{3}{11}$, $\frac{9}{11}$, $\frac{14}{11}$, etc.

3. Can you find a decimal numeral for $\frac{5}{11}$ without dividing?
4. Can you find a decimal for $\frac{7}{11}$ without dividing?
5. Is it true that the decimal numeral for $\frac{3}{11}$ is three times that of $\frac{1}{11}$?

Exercises 8-6c

1. Find decimal numerals for the following numbers.

(a) $\frac{1}{2}$
(b) $\frac{1}{5}$
(c) $\frac{3}{4}$
(d) $\frac{3}{5}$
(e) $\frac{6}{10}$
(f) $\frac{1}{8}$
(g) $\frac{4}{5}$
(h) $\frac{3}{8}$
(i) $\frac{2}{3}$
(j) $\frac{1}{11}$
(k) $\frac{5}{9}$
(l) $\frac{3}{17}$

2. Which decimal numerals in problem 1 end?
3. Which decimal numerals in problem 1 repeat but do not end?
4. Find decimal numerals for the first number in each group and calculate the others without dividing.

(a) $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$
(b) $\frac{1}{10}$, $\frac{4}{10}$, $\frac{7}{10}$
(c) $\frac{1}{8}$, $\frac{3}{8}$, $\frac{7}{8}$
(d) $\frac{1}{11}$, $\frac{5}{11}$, $\frac{9}{11}$
(e) $\frac{1}{20}$, $\frac{3}{20}$, $\frac{11}{20}$
(f) $\frac{1}{50}$, $\frac{42}{50}$, $\frac{47}{50}$
(g) $\frac{1}{1000}$, $\frac{112}{1000}$, $\frac{927}{1000}$

We have found a decimal numeral for $\frac{1}{8}$. We said 0.125 was a terminating decimal numeral because at some stage in the division process we obtained a zero remainder.

Let us reconsider a decimal numeral for $\frac{1}{8}$. Recall it was obtained by dividing 1 by 8.

```
    0.125
8 )1.000
   8
   20
   16
    40
    40
     0
```

After the first subtraction the remainder is 2.
the second remainder is 4.
the third remainder is 0.

We stopped or terminated our division process at the stage when we obtained a zero remainder. However we could just as well have continued dividing getting at each new stage a remainder of zero (and a quotient of zero).

It is clear that once we get a remainder of zero every remainder thereafter will be zero.

Let us pause for a moment and consider what we mean when we say that a decimal numeral is a repeating decimal. At some decimal place a digit or block of digits begins to repeat and continues to repeat in a cycle over and over again without end. We see that the decimal numeral 0.125000... for $\frac{1}{8}$ fits these conditions. From the fourth decimal place on, the digit zero repeats over and over again without end. Therefore 0.125000... is a repeating decimal just as 0.333... is a repeating decimal. In 0.333... the digit 3 repeats over and over again.

We have referred to the two examples 0.125000... and 0.333... They are usually called the decimal expansions of $\frac{1}{8}$ and $\frac{1}{3}$

respectively. By decimal expansion we mean that there is a digit for every decimal place. Note that the decimal expansion of $\frac{1}{8}$ is repeating and also terminating. The decimal expansion of $\frac{1}{3}$ is repeating but not terminating.

You have seen that every rational number may be named by a fraction $\frac{a}{b}$, where a is a whole number and b is a counting number. You have also seen that we have a procedure for finding a decimal numeral to name the rational number $\frac{a}{b}$.

We now have the answer to our first question: "May every rational number be named by some decimal numeral?" Not only do we know that we may find some decimal numeral to name the rational number, but we also see that in all the exercises we have worked, the decimal numerals naming rational numbers either end ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, etc.) or repeat ($\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, $\frac{3}{11}$, etc.) after some place to the right or to the left of the decimal point.

Here are some more samples:

(a) $\frac{200}{2} = 100.000\ldots$

(b) $\frac{2000}{9} = 222.222\ldots$

(c) $\frac{1}{7} = .142857\overline{142857}\ldots$

(d) $\frac{743,000,001,000,011}{999,000,000,000,000} = .743743744744755755\overline{755}\ldots$

(e) $\frac{245125}{1000} = 245.125\overline{0}00\ldots$

Do the decimal expansions for rational numbers all either end or repeat? Let's look back at the procedure by which we found a decimal numeral for $\frac{1}{7}$.

```
      .142857142857...
  7 )1.0000000000000...
     7
    --
     30
     28
     --
      20
      14
      --
       60
       56
       --
        40
        35
        --
         50
         49
         --
          10
           7
          --
           30
           28
           --
            20
            14
            --
             60
             56
             --
              40
              35
              --
               50
               49
               --
                10
```

After the first subtraction the remainder is 3

The second remainder is 2

The third remainder is 6

The fourth remainder is 4

The fifth remainder is 5

The sixth remainder is 1

The seventh remainder is 3 <u>again</u>

The eighth remainder is 2 again

Class Discussion Questions

1. List the (a) 9th, (b) 10th, (c) 11th, and (d) 12th remainders. Observe that the decimal numeral repeats as the remainder repeats.

2. Can you tell why this must happen?

3. Why is it that the remainder can never be 7 or more? What error would we have made in the division if, at some stage our remainder was equal or greater than 7?

The only remainders we could have are 0, 1, 2, 3, 4, 5, 6. If the remainder at some stage were zero the decimal numeral would end or the digit zero repeats. If the remainder is never zero as in the example above, it can only be 1, or 2, or 3, or 4, or 5, or 6.

Therefore, after no more than 6 steps, a remainder will have to appear which has appeared before at some earlier stage. The next step in the process, then, is to repeat a division which has been done earlier, so the whole process begins to repeat. Remainders and digits in the quotient begin repeating at this stage. Notice that the repetition may begin after fewer than 6 steps since it will begin whenever a number appears as a remainder for the second time.

In finding the decimal numeral for $\frac{1}{3}$, you know that the remainders can be 0, 1, 2. The remainder is never zero, so the decimal numeral doesn't end. As a matter of fact the remainder is never two either. The first remainder is one and the repeating begins in the very next step.

4. How many remainders could we have if we were dividing 3 by 17 to find a decimal numeral naming $\frac{3}{17}$?
5. If a remainder is zero how do we describe the decimal numeral?
6. If the decimal numeral does not end, how can we be sure it will repeat?

7. What is the maximum number of remainders we would have to write before the decimal numeral would have to repeat?

8. May the decimal numeral repeat before we have written the first 16 remainders?

Hence every rational number may be named by a decimal numeral which either ends or repeats.

Exercises 8-6d

1. Write decimal numerals for the following numbers:

   (a) $\frac{2}{3}$

   (b) $\frac{5}{8}$

   (c) $\frac{3}{11}$

   (d) $\frac{5}{12}$

   (e) $\frac{13}{45}$

   (f) $\frac{24}{105}$

2. Write decimal numerals for the following numbers:

   (a) $\frac{1}{9}$

   (b) $\frac{2}{9}$

   (c) $\frac{3}{9}$

   (d) $\frac{4}{9}$

   (e) $\frac{5}{9}$

   (f) $\frac{6}{9}$

   (g) $\frac{7}{9}$

   (h) $\frac{8}{9}$

   (i) $\frac{9}{9}$

   What conclusion can be drawn about the answer to (i)?

3. Write decimal numerals for the following numbers:

   (a) $\frac{1}{7}$

   (b) $\frac{2}{7}$

   (c) $\frac{3}{7}$

   (d) $\frac{4}{7}$

   (e) $\frac{5}{7}$

   (f) $\frac{6}{7}$

   (g) $\frac{7}{7}$

*4. Show that: The rational number $\frac{a}{b}$ may be named by a decimal numeral which ends, if b has only 2's or 5's or both as prime factors. Hint:

(a) Is a rational number whose fraction name has a denominator which is a power of 2 named by a terminating decimal numeral?

(b) Is a rational number whose fraction name has a denominator which is a power of 5 named by a terminating decimal numeral?

(c) Is a rational number whose fraction name has a denominator which is a product of powers of 2 and 5 named by a terminating decimal numeral?

(d) Is a rational number whose fraction name has a denominator which is not a power of 2, or a power of 5, or a product of powers of 2 and 5, named by a terminating decimal numeral?

*5. BRAINBUSTER: Show: The rational number $\frac{a}{b}$ may be named by a decimal numeral which either ends or repeats after (b - 1) remainders at most. (Hint: Can you re-write such a fraction as one whose denominator is a power of 10?)

Now let us consider the second question: "Does every decimal expansion name a rational number?" Can you think of a decimal expansion that does not end and does not repeat no matter how many digits you write?

Consider the following decimal expansion:

.010010001000010000010000001...

Can you discover a law of formation?

Does it end?

Does it repeat?

We now can guess the answer to our second question: Not every decimal expansion names a rational number.

If it were convenient for purposes of calculation to use a fraction rather than a repeating decimal numeral, could we find one?

Name: (a) 0.333... by a fraction

(b) 3.333... by a fraction.

Is 3.333... ten times as large as 0.333...?

What is the difference between 3.333... and 0.333...?

$$\frac{1}{3} = 0.333\ldots$$

$$3\frac{1}{3} = 3.333\ldots$$

$$3\frac{1}{3} - \frac{1}{3} = 3.333\ldots - 0.333\ldots$$

$$3 = 3$$

Let us suppose we did not know that $\frac{1}{3} = 0.333\ldots$

Let us call the rational number it names x.

$$\begin{aligned} \text{Then} \quad x &= 0.333\ldots \\ 10x &= 3.333\ldots \\ 10x - x &= 3.333\ldots - 0.333\ldots \\ 9x &= 3 \\ x &= \frac{3}{9} \\ x &= \frac{1}{3} \end{aligned}$$

Let us develop a similar procedure for finding the rational number named by 0.666...

$$\begin{aligned} x &= 0.666\ldots \\ 10x &= 6.666\ldots \\ 10x - x &= 6.666\ldots - 0.666\ldots \\ 9x &= 6 \\ x &= \frac{6}{9} \\ x &= \frac{2}{3} \end{aligned}$$

Example:

(a) Name $0.1212\overline{12}\ldots$ by a fraction.

(b) Name $1.212\overline{12}\ldots$ by a fraction.

Solution: The above examples illustrate a possible procedure which we may follow.

(a)

$$\begin{aligned} x &= 0.1212\overline{12}\ldots \\ 100x &= 12.1212\overline{12}\ldots \end{aligned}$$

Why did we multiply by 100 this time?

$$\begin{aligned} 100x - x &= 12 \\ 99x &= 12 \\ x &= \frac{12}{99} = \frac{4}{33} \end{aligned}$$

(b)

$$\begin{aligned} y &= 1.212\overline{12}\ldots \\ 100y &= 121.212\overline{12}\ldots \\ 100y - y &= 120 \\ 99y &= 120 \\ y &= \frac{120}{99} = \frac{40}{33} \end{aligned}$$

Perform the division to check that the decimal expansion of $\frac{40}{33}$ is 0.121212...

Exercises 8-6e

Name the following numbers by fractions.

(a) $0.1313\overline{13}\ldots$

(b) $0.4545\overline{45}\ldots$

(c) $0.833\overline{3}\ldots$

(d) $0.8484\overline{84}\ldots$

(e) $7.33\overline{3}\ldots$

(f) $9.66\overline{6}\ldots$

(g) $4.1212\overline{12}\ldots$

(h) $2.2424\overline{24}\ldots$

*(i) $1.5162\overline{5162}\ldots$

*(j) $4.714284\overline{714285}\ldots$

Check the answer you obtain in each case by performing the division.

8-7. The Metric System

In Chapter 2 you studied systems of numeration. You will recall that human beings, after many years of struggle with various systems, finally developed the Hindu-Arabic numerals and the base 10 or decimal system of notation.

In the year 1585 a Flemish mathematician named Simon Stevin made one of the simplest yet cleverest contributions of all time. He took an ordinary dot (.), placed it to the right of a whole number and made a symbol to indicate that the position to the immediate left of the point is the ones place while the position to the immediate right is the tenths place. The decimal point extended the "tens" system to the right as well as to the left. It made it possible to use positional notation for fractions as well as whole numbers.

In the year 1789 a group of French mathematicians and scientists were called together to develop a simplified system of weights and measures. This system was to be used throughout the new France created by the French revolution. They decided to take full advantage of the Hindu-Arabic numerals, the decimal system of notation, and Simon Stevin's decimal point. They planned and measured and calculated. Finally, they decided upon the meter as the unit of length. It was intended to be $\frac{1}{20,000,000}$ of the meridian through Paris. (A meridian is a half-circle on the earth's surface joining the north and south poles.) In other words, the distance from the north pole to the south pole when measured on the earth's surface was taken to be 20,000,000 meters.

On June 22, 1799, a platinum meter was adopted as the true meter, and was deposited in the Archives of the State, where it has become known as the Meter of the Archives.

We now know that errors crept into the calculations. The computations were revised in 1927, and the standard meter was then defined so that:

The Hindu-Arabic numerals, the base 10, and the decimal point were used in the following way:

They subdivided the meter into 10 equal parts calling the length of each part a decimeter of $\frac{1}{10}$ of a meter. Each decimeter was divided into 10 equal parts. The length of these parts they called a centimeter ($\frac{1}{10} \times \frac{1}{10}$ or $\frac{1}{100}$ of a meter). Each centimeter was divided into 10 equal parts. The length of these parts they called a millimeter ($\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}$ or $\frac{1}{1000}$) of a meter. For longer distances they used 10 meters called a dekameter (10 × 1 meter), a hectometer (10 × 10 or 100 meters), and a kilometer (10 × 10 × 10 or 1000 meters).

The following table shows their plan in an orderly form (the starred entries represent those units which are most frequently used).

Table 1

$10 \times 10 \times 10 = 1000$ thousand	$10 \times 10 = 100$ hundred	10 ten	1 unit	$\frac{1}{10}$ tenth	$\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ hundredth	$\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1000}$ thousandth	decimal notation for length
*kilometer (km)	hectometer (hm)	dekameter (dkm)	*meter (m)	decimeter (dm)	*centimeter (cm)	*millimeter (mm)	metric measurement

Exercises 8-7a

1. Write in expanded form each of the following:

Sample: 1234.567 meters = 1 kilometer + 2 hectometers + 3 dekameters + 4 meters + 5 decimeters + 6 centimeters + 7 millimeters or 1 km + 2 hm + 3 dkm + 4m + 5 dm + 6 cm + 7 mm

(a) 3030.303 meters

(b) 245.36 m

(c) 5.342 m

(d) 0.564 m

(e) 0.043 m

2. Complete each of the following:

(a) 1 kilometer = __________ hectometers

(b) 1 kilometer = __________ dekameters

(c) 1 kilometer = __________ meters

(d) 1 kilometer = __________ decimeters

(e) 1 kilometer = __________ centimeters

(f) 1 kilometer = __________ millimeters

3. Complete each of the following:

(a) $1\frac{1}{2}$ km = __________ m

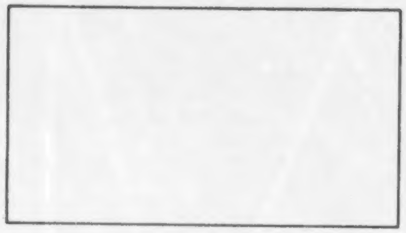
(b) $3\frac{1}{4}$ m = __________ cm

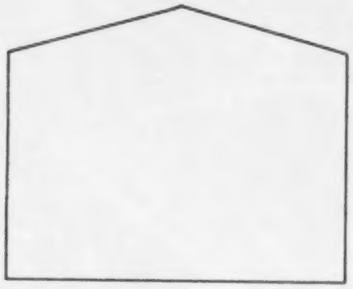
(c) 2.54 cm = __________ mm

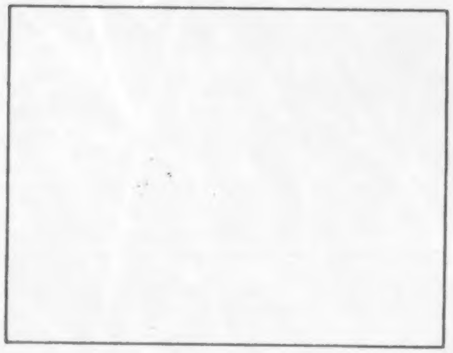
(d) 200 m = __________ cm

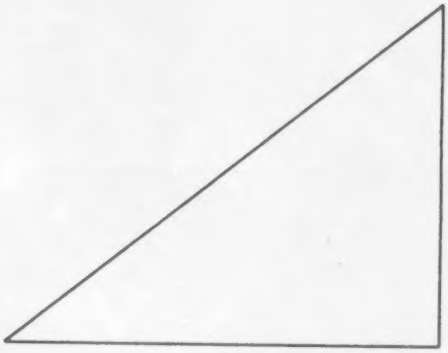
(e) 500 m = __________ km

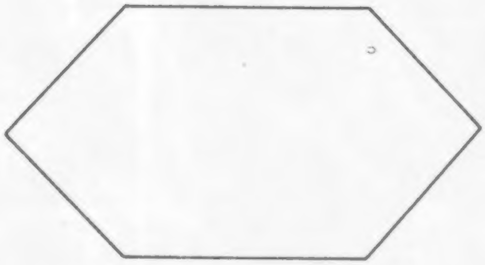
4. Measure the boundary line of these closed curves. The measurement is to be made to the nearest centimeter.

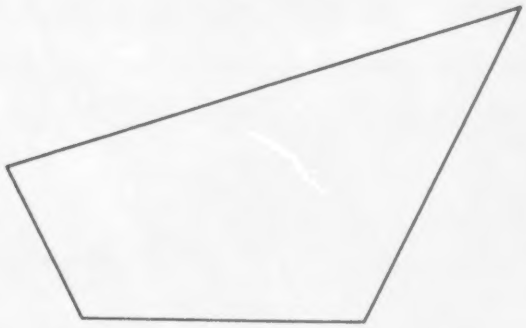












5. Measure the boundary lines of the same closed figures. Each measurement is to be made to the nearest millimeter.

6. Which is the more precise unit of measure, the centimeter or the millimeter?

7. The measurement of the line segment below is one decimeter.

   (a) Use your ruler to help you divide it into 10 equal parts.

   (b) What is the measurement of each part called?

   (c) Divide one of these parts into 10 equal parts.

   (d) What is the measurement of each of the parts called?

   (e) What is the measurement of a line segment 10 times as long as the decimeter called?

8. Use your ruler to draw line segments each of whose measurement is:

   (a) 2.54 cm

   (b) 10 mm

   (c) 5 cm

   (d) 25.4 mm

   (e) 50 mm

   (f) 10 cm

   (g) 100 mm.

*9. If a meter is defined to be $\frac{1}{10,000,000}$ of the distance on the earth's surface from the north pole through Paris to the equator

   (a) How many meters are in the circumference of the earth?

   (b) How many dekameters are in the circumference of the earth?

   (c) How many hectometers are in the circumference of the earth?

(d) How many kilometers are in the circumference of the earth?

(e) How many decimeters are in the circumference of the earth?

(f) How many centimeters are in the circumference of the earth?

(g) How many millimeters are in the circumference of the earth?

Research problem:

(a) The micron is a unit of measure which is used to measure bacteria. Smaller units are called the $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ of a micron and are used to measure viruses, components of cells and large molecules. Some of these units also serve to measure the wavelengths of some radiations. What fraction of a centimeter is a micron?

(b) Atomic measurements required the use of an approximate unit of length: The Angstrom (abbreviation A). For instance, the dimensions of the atom are of that size, while the nucleus has a radius of $\frac{1}{100,000}$ of an A. What fraction of a centimeter is an Angstrom?

(c) The meter is now defined in a new way. The new definition appears on page 83 of the twentieth yearbook of the National Council of Teachers of Mathematics. Look up the definition and report it to your class.

(d) The early history of the metric system is described in the above reference book. Prepare a report for the class based upon your reading of pages 22-33 of this reference book.

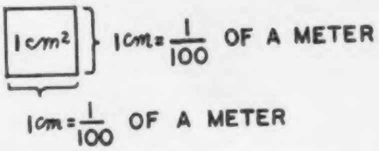
Now that you are familiar with the various units of metric linear measurement, let us study the metric units of area, volume, weight, and capacity.

In Chapter 7 you were asked to find the areas of the interior of a variety of simple closed curves. You were also asked to find the volumes of the interior of various rectangular solids. You recall you finally chose a square region as the best unit of measure for the area of the interior of the closed curves. The interior of a cube was chosen as the unit of measure for the volume of the interior of the rectangular solids.

The metric units for measuring areas is also a square region. The measurement of the edges of this square is one meter. The area of the interior of this square region is one square meter (abbreviation 1 m^2).

Figure 1 below is a picture of one square centimeter (1 cm^2).

Figure 1



$$1\,cm^2 = \frac{1}{100}\,m \times \frac{1}{100}\,m = \frac{1}{10{,}000}\,m^2$$

We can now show the multiples and subdivisions of the square meter.

Table 2

Length	*km (1000 m)	hm (100 m)	dkm (10 m)	*m	dm ($\frac{1}{10}$ m)	*cm ($\frac{1}{100}$ m)	*mm ($\frac{1}{1000}$ m)
Area	*km^2 (1000^2m^2)	hm^2 (100^2m^2)	dkm^2 (10^2m^2)	*m^2	dm^2 ($\frac{1}{10^2}m^2$)	*cm^2 ($\frac{1}{100^2}m^2$)	*mm^2 ($\frac{1}{1000^2}m^2$)

Exercises 8-7b

1. Complete each of the following:

Sample: There are 10^2 or ___100___ m^2 in 1 dkm^2

(a) There are 100^2 or ________ m^2 in 1 hm^2.

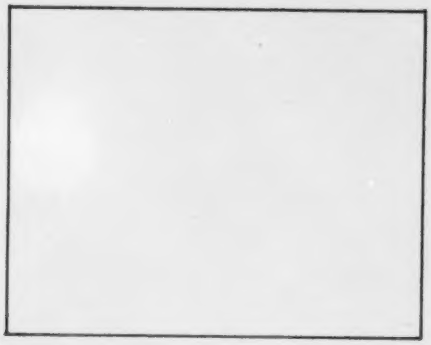
(b) There are 1000^2 or ________ m^2 in 1 km^2.

(c) There are $(\frac{1}{10})^2$ or ________ m^2 in 1 dm^2.

(d) There are $(\frac{1}{100})^2$ or ________ m^2 in 1 cm^2.

(e) There are $(\frac{1}{1000})^2$ or ________ m^2 in 1 mm^2.

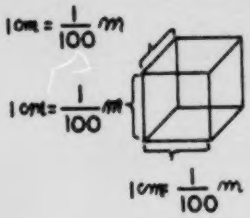
2. Use your ruler to find the measurement of the length and width of the rectangle below and find the area. Make all measurements to the nearest cm.



3. Use your ruler to find the measurement of the length and width of the rectangle above and calculate the area. Make all measurements to the nearest mm.

The metric unit of measure for measuring volume is also the interior of a cube. The measurement of each edge of this cube is 1 meter. The volume of the interior of this cube is 1 cubic meter (abbreviation 1 m^3). Figure two below is a picture of 1 cubic centimeter (1 cm^3).

Figure 2


1cm = 1/100 m
1cm = 1/100 m
1cm = 1/100 m

$$1\,cm^3 = \frac{1}{100}\,m \times \frac{1}{100}\,m \times \frac{1}{100}\,m = \frac{1}{1{,}000{,}000}\,m^3$$

We can show the multiples and subdivisions of the cubic meter in Table 3.

Table 3

Decimal notation

Length	*km (1000 m)	hm (100 m)	dkm (10 m)	*m	dm ($\frac{1}{10}$ m)	*cm ($\frac{1}{100}$ m)	*mm ($\frac{1}{1000}$ m)
Area	*km^2 = $(1000)^2 m^2$	hm^2 = $(100)^2 m^2$	dkm^2 = $(10)^2 m^2$	*m^2	dm^2 = $(\frac{1}{10})^2 m^2$	*cm^2 = $(\frac{1}{100})^2 m^2$	*mm^2 = $(\frac{1}{1000})^2 m^2$
Volume	*$(1000)^3 m^3$ = km^3	$(100)^3 m^3$ = hm^3	$(10)^3 m^3$ = dkm^3	*m^3	$(\frac{1}{10})^3 m^3$ = dm^3	*$(\frac{1}{100})^3 m^3$ = cm^3	*$(\frac{1}{1000})^3 m^3$ = mm^3

Exercises 8-7c

1. Complete each of the following:

Sample: There are 10^3 or ___1000___ m^3 in 1 dkm^3.

(a) There are 100^3 or __________ m^3 in 1 hm^3.

(b) There are 1000^3 or __________ m^3 in 1 km^3.

(c) There are $(\frac{1}{10})^3$ or __________ m^3 in 1 dm^3.

(d) There are $(\frac{1}{100})^3$ or __________ m^3 in 1 cm^3.

(e) There are $(\frac{1}{1000})^3$ or __________ m^3 in 1 mm^3.

2. The rectangle measured in problems 2 and 3 of Exercises 8-7b is the base of a rectangular solid. The measure of the height to the nearest cm is 6 cm. Calculate the volume of the interior of this rectangular solid. Remember the volume of the interior of a rectangle solid is equal to the product of the number of units of measure in the length, width and height.

3. The measure of the height of the rectangular solid mentioned in the above problem is 59 mm to the nearest mm. Calculate the volume of the interior of this rectangular solid using the area of the rectangle calculated in problem 3 of Exercises 8-7b.

The metric unit for the measure of weight was intended to be the weight of the water contained by a vessel whose volume is one cubic centimeter. The weight of 1 cubic centimeter of water is called a gram. Thus, the instant we know the volume of the interior of a container we immediately know the weight of water it can contain. For example if the volume of the interior of a container is 500 cm^3, then the weight of water it can contain is 500 grams. 500 grams $\approx$ 1 pound.

Table 4

Length	*km	hm	dkm	*m	dm	*cm	*mm
Area	*$(1000)^2m^2$ = km^2	$(100)^2m^2$ = hm^2	$(10)^2m^2$ = dkm^2	*m^2	$(\frac{1}{10})^2m^2$ = dm^2	*$(\frac{1}{100})^2m^2$ = cm^2	*$(\frac{1}{1000})^2m^2$ = mm^2
Volume	*$(1000)^3m^3$ = km^3	$(100)^3m^3$ = hm^3	$(10)^3m^3$ = dkm^3	*m^3	$(\frac{1}{10})^3m^3$ = dm^3	*$(\frac{1}{100})^3m^3$ = cm^3	*$(\frac{1}{1000})^3m^3$ = mm^3
Weight	*kgm (1000 gm)	hgm (100 gm)	dkgm (10 gm)	*gm	dgm ($\frac{1}{10}$ gm)	*cgm ($\frac{1}{100}$ gm)	*mgm ($\frac{1}{1000}$ gm)

We can now show the multiples and subdivisions of the gram.

Exercises 8-7d

1. Write in expanded form:

   (a) 4444.444 gms

   (b) 567.8 gms.

2. What is the weight of water (in grams) that can be contained by the rectangular solid of problem 2 of Exercises 8-7c?

3. What is the weight of water (in kgms) that can be contained by the rectangular solid of problem 2 of Exercises 8-7c?

Capacity is another name for volume, but it refers to those vessels that are supposed to contain something, while volume refers to any portion of three dimensional space.

A box is a rectangular solid. The volume of its interior can be found as in problems 2 and 3 of Exercises 8-7c. To speak of its capacity is to hint at what it can contain.

In everyday life in America, we say the capacity of a tank is so many gallons, its volume is so many cubic feet, and the weight of its contents is so many pounds. It is usually customary to talk of the capacity of a milk bottle as a quart or two pints and not the weight of the milk or the volume of the milk. To talk of capacity we use a special unit of measure.

In the metric system the unit of capacity is the liter. It is defined to be the capacity of a cube whose edges each measure 10 centimeters or 1 decimeter. Thus a cube whose edges measure 1 decimeter has a volume of 1000 cm^3. It can contain a weight of 1000 grams of water and its capacity is 1 liter. 1 liter is about one quart.

We can now show the multiples and subdivisions of the liter.

Table 5

Length	*km	hm	dkm	*m	dm	*cm	*mm
Area	*$(1000)^2 m^2 = km^2$	$(100)^2 m^2 = hm^2$	$(10)^2 m^2 = dkm^2$	*m^2	$(\frac{1}{10})^2 m^2 = dm^2$	*$(\frac{1}{100})^2 m^2 = cm^2$	*$(\frac{1}{1000})^2 m^2 = mm^2$
Volume	*$(1000)^3 m^3 = km^3$	$(100)^3 m^3 = hm^3$	$(10)^3 m^3 = dkm^3$	*m^3	$(\frac{1}{10})^3 m^3 = dm^3$	*$(\frac{1}{100})^3 m^3 = cm^3$	*$(\frac{1}{1000})^3 m^3 = mm^3$
Weight	*kgm 1000 gms	hgm 100 gms	dkgm 10 gms	*gm	dgm $\frac{1}{10}$ gm	*cgm $\frac{1}{100}$ gm	*mgm $\frac{1}{1000}$ gm
Capacity	*kl 1000 l	hl 100 l	dkl 10 l	*l	dl $\frac{1}{10}$ l	*cl $\frac{1}{100}$ l	*ml $\frac{1}{1000}$ l

The table above clearly shows the use of the decimal system of notation. However it does not conveniently reveal the 1 to 1 correspondence between the units of measure for volume, weight, and capacity.

Table 6 below shows this 1 to 1 correspondence more clearly

Table 6

Volume	Weight of Water	Capacity
1 cubic centimeter	1 gram	1 milliliter
1 cubic decimeter (1000 cm^3)	1 kilogram (1000 grams)	1 liter (1000 milliliters)
1 cubic meter (1000 cubic decimeters)	1 metric ton	1 kiloliter 1(1000 liters)

Exercises 8-7e

1. Write in expanded form:
   (a) 6666.666 liters
   (b) 2.03 liters
2. What is the capacity (in milliliters) of the rectangular solid mentioned in problem 2 of Exercises 8-7c?
3. What is the capacity in liters of the rectangular solid mentioned in problem 2 of Exercises 8-7c?

Now that you have lived with the metric system of weights and measures, perhaps you might be interested in converting from the metric system to the English system. Table 7 gives a conversion list of the more commonly used measures.

Table 7

Metric		English
2.54 centimeters	=	1 inch
1 meter	≈	1.1 yards
1 kilometer	≈	$\frac{3}{5}$ or .6 miles
1 kilogram	≈	2.2 pounds
1 metric ton	≈	1.1 tons
1 liter	≈	1 quart

Exercises 8-7f

1. Convert each of the following metric measurements to equivalent English measurements. (You may refer to the table of conversions, if necessary.)

(a) 25.4 cm = __________ in. (the length of the usual slide rule)

(b) 100 meters ≈ __________ yds.
(c) 200 meters ≈ __________ yds.
(d) 400 meters ≈ __________ yds.
(e) 800 meters ≈ __________ yds.
(f) 1500 meters ≈ __________ yds.

A few of the standard distances for track and field events.

(g) 10 kilometers ≈ __________ miles
(h) 100 kilometers ≈ __________ miles.
(i) 10 kilograms ≈ __________ lbs.
(j) 100 kilograms ≈ __________ lbs.
(k) 4 liters ≈ __________ quarts ≈ __________ gallons.
(l) 60 liters ≈ __________ quarts ≈ __________ gallons.
(m) 10 metric tons ≈ __________ American tons ≈ __________ lbs.
(n) 100 metric tons ≈ __________ American tons ≈ __________ lbs.

2. (a) Find the volume in cubic inches and in cubic feet (nearest 10th) of a cubical tank 6 ft. 9 inches each way. If the tank is filled with water find its weight. Time yourself.
   Note: 1728 cubic inches = 1 cubic foot.
   1 cubic foot of water weighs 62.4 lbs.

   (b) Calculate the volume in meters and the weight in kilograms of a cubical tank 2.07 meters each way, filled with water. Time yourself. (Note: There are 1000 liters in a cubic meter. 1000 kilograms is the weight of 1000 liters.)

   (c) Compare the time needed to solve problem (a) with the time needed to solve problem (b). Which problem required more of your time?

Research problem:

Use the Twentieth Yearbook of the National Council of Teachers of Mathematics as your reference book.

(a) Collect a list of all the various units of measurement for length, area, volume, and capacity that you can find listed in your reference book. Bring this list to school.

(b) Write a composition on "Why I Prefer the English System of Weights and Measures to the Metric System" or "Why I Prefer the Metric System to the English System". You may use your reference book to aid you in getting information.

*3. Which weighs more, a pound of gold or a pound of feathers?

To summarize our work with the metric system we present the following table: (From this table you can derive all of the multiples and subdivisions we presented in Table 6).

Table 8

	Length	
1 centimeter (cm)	=	10 millimeters (mm)
1 meter (m)	=	100 centimeters (cm)
1 kilometer (km)	=	1000 meters (m)
	Capacity	
1 liter (l)	=	1000 milliliters
	Weight	
1 kilogram	=	1000 grams (g)
1 metric ton (t)	=	1000 kilograms (kg)

8-8. Summary

1. The *ratio* of a number a to a number b $(b \neq 0)$ is the quotient $\frac{a}{b}$.
2. Two physical quantities are said to be *proportional* to one another if the ratio of their measures is always the same.
3. *Property 1*. Any fraction, $\frac{a}{b}$, can be expressed as a *percent* by finding c such that $\frac{a}{b} = \frac{c}{100} = c \times \frac{1}{100} = c\%$.
4. A rational number $\frac{a}{b}$ can be written in decimal form by dividing a by b.
5. *Property 2*. To find the number of decimal places when two numbers are multiplied, add the number of decimal places in the two numbers.
6. A commission is the payment, often based on a percent of selling price, that is paid to a salesman for his service.
7. A discount is the amount subtracted from the marked price.

8. Sale price or net price is the marked price less the discount.

9. A decimal numeral ends or terminates if it is obtained by a division which is exact, or in which there occurs a remainder of zero.

10. A decimal numeral is a repeating decimal or repeats if, from some decimal place, a digit or block of digits repeats.

11. Every rational number may be named by a decimal numeral which either terminates or repeats.

12. Conversion Table

Metric		English
2.54 centimeters	=	1 inch
1 meter	≈	1.1 yards
1 kilometer	≈	0.6 miles
1 kilogram	≈	2.2 pounds
1 metric ton	≈	1.1 tons
1 liter	≈	1 quart

13. Table of Metric Units

1 centimeter (cm) = 10 millimeters (mm)
1 meter (m) = 100 centimeters
1 kilometer (km) = 1000 meters
1 liter (l) = 1000 milliliters (ml)
1 kilogram (kg) = 1000 grams (g)
1 metric ton = 1000 kilograms

CHAPTER 9

PARALLELS, PARALLELOGRAMS, TRIANGLES, AND RIGHT PRISMS

9-1. Two Lines in a Plane

Geometry had an influence on the way in which man lived as early as 4000 B.C. Clay tablets made in Babylon some 6000 years ago show that Babylonians knew how to find the area of a rectangular field. They used the same relation you learned in Chapter 7.

The great pyramid at Giza in Egypt was built about 2900 B.C. The constructions of this and other pyramids show that the Egyptians too must have known much about geometry. About 1850 B.C., Egyptian manuscripts were being written which show that the Egyptians were developing geometric rules. One of these was the general rule for finding the area of a triangle.

When a mathematician begins an investigation, he usually starts with a very simple case. After he feels that he understands this case, he may then proceed to more complicated situations. In order to get a feeling for spatial relationships,let us begin by studying more about figures formed by two lines.

Figure 9-1-a shows that the intersection of lines ℓ_1 and ℓ_2 is point A. Rays on ℓ_2 are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (Recall that $\overrightarrow{AB}$ means the ray with A as endpoint and containing point B.) Rays on ℓ_1 are $\overrightarrow{AE}$ and $\overrightarrow{AD}$.

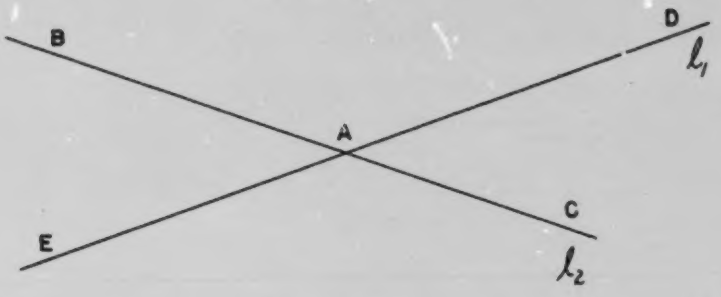


Figure 9-1-a

Observe that there are four angles formed by the rays in Figure 9-1-a: angle BAE, angle BAD, angle CAD, and angle CAE. We shall speak of these angles as angles determined by ℓ_1 and ℓ_2. In this chapter, then, we shall speak of two intersecting lines as determining four angles.

Look at angles CAD and DAB. These two angles have a common ray, AD, and a common vertex, point A. Any two angles which have a common ray, a common vertex, and whose interiors have no point in common are called adjacent angles. (Adjacent means "neighboring.") Thus, angles CAD and DAB are adjacent angles. Are there any more pairs of adjacent angles in the figure? Yes: DAB and BAE, BAE and CAE, CAE and CAD, are all pairs of adjacent angles.

Are angles BAD and CAE adjacent angles? No, they are not! But they are both formed by rays on the two lines ℓ_1 and ℓ_2. In this chapter, when we speak of a line we will always mean a straight line, extending without end in both directions. When two lines intersect, the two pairs of non-adjacent angles determined by these lines are given a special name, vertical angles. Note that here "vertical" is not associated with "horizontal." Thus, angles BAE and CAD are a pair of vertical angles.

In Figure 9-1-a, consider the line $\overleftrightarrow{BC}$ containing the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Recall that in Chapter 7 you learned that if we place a protractor with vertex at A so that $\overrightarrow{AC}$ is the zero ray, then $\overrightarrow{AB}$ corresponds to the number 180 on a protractor.

A line divides a plane into two half planes. In Figure 9-1-b, the two half planes determined by ℓ_1 are shown by the shaded area above ℓ_1 and the non-shaded area below ℓ_1.

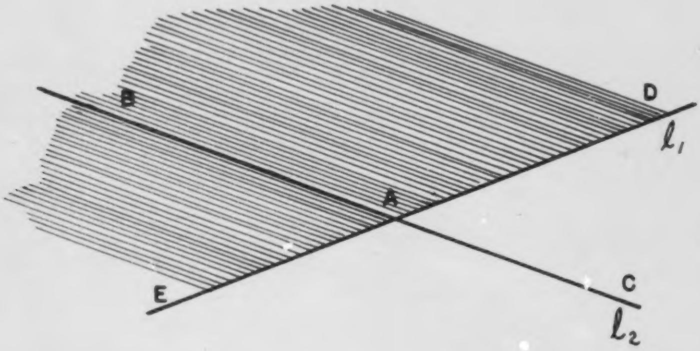


Figure 9-1-b

In Figure 9-1-c, the two half planes determined by ℓ_2 are shown by the shaded area below ℓ_2 and the non-shaded area above ℓ_2.

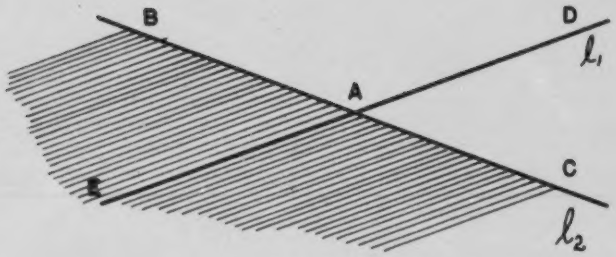


Figure 9-1-c

Thus, two intersecting lines separate a plane into four regions. The interior of one of a pair of vertical angles is the intersection of two of these four half planes. Figure 9-1-d shows the interior of angle BAE as the intersection of the two half planes described above. The interior of angle CAD, the other angle in the pair of vertical angles, is the intersection of the two remaining half planes. Can you see the two half planes which include the interior of angle CAD?

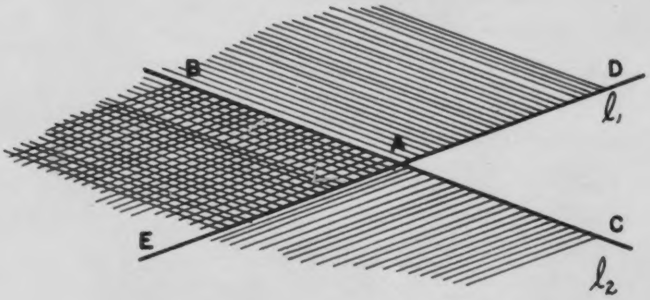


Figure 9-1-d

Two angles whose measures in degrees add up to 180 are called supplementary angles. In Figure 9-1-a, angles CAE and CAD are supplementary. Are any other pairs of angles in this figure supplementary? Yes: $\angle$CAD and $\angle$BAD; $\angle$BAD and $\angle$BAE are pairs of supplementary angles. Can you find any other pairs of supplementary angles? In Figure 9-1-a the supplementary angles are also adjacent angles. In Figure 9-1-e, the measure of angle M is 40 and the measure of angle N is 140. Angles M and N are supplementary angles.

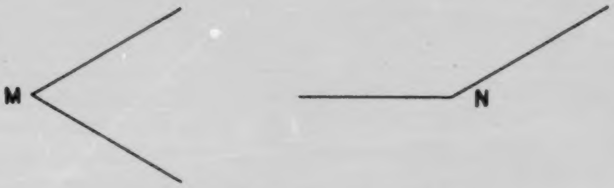


Figure 9-1-e

You learned in Chapter 7 to measure angles. We shall frequently talk about the "measure of an angle" in this chapter so it is convenient to have a symbol for this statement. To indicate the number of units in an angle, we will use the symbol "m" followed by the name of the angle enclosed in parentheses. For example, m($\angle$ABC) means the number of units in angle ABC.

As we learned, any angle can be used as a unit of measure, but in our work so far the degree has been the only standard unit. We will use the degree as the standard unit unless something else is specified. Thus when we write m($\angle$ABC) = 40, we will understand angle ABC is a 40 degree (40°) angle. Note that since m($\angle$ABC) is a number, we write only m($\angle$ABC) = 40, not "m($\angle$ABC) = 40°."

Exercises 9-1

Use Figure 9-1-f in answering 1 through 3.

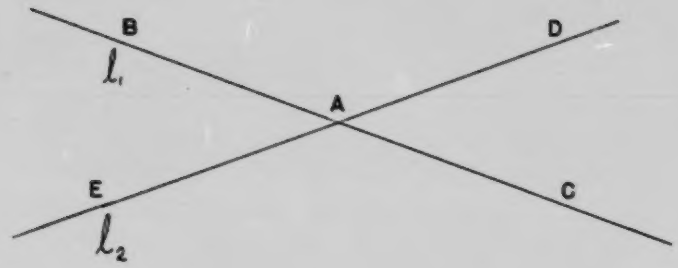


Figure 9-1-f

1. (a) Name the angles adjacent to ∠ BAE.

   (b) Name the angles adjacent to ∠ DAC.

   (c) Name the angles adjacent to ∠ BAD.

2. (a) Name the angle which with ∠ BAE completes a pair of vertical angles.

   (b) Name another pair of vertical angles in the figure.

   (c) When two lines intersect in a point, how many pairs of vertical angles are formed?

3. (a) Use a protractor to find the measures of the vertical angles, ∠ BAE and ∠ DAC.

   (b) What are the measures of angles CAE and BAD?

   (c) What appears to be true concerning the measures of a pair of vertical angles?

   (d) Draw sets of two intersecting lines as in Figure 9-1-f. Vary the size of the angles between the lines. With a protractor find the measures of the interiors of each pair of vertical angles. Do they appear equal?

4. Study your answers to Problem 3. Then state your results in the form of a general property, by copying and completing the following sentence:

   Property 1: When two lines intersect, the two angles in each pair of ____?____ angles which are formed have __?__ measure.

*5. You have found by experiment a certain relation to be true in a number of cases. Let us see why this relation must be true in all cases.

   (a) In Figure 9-1-f what is the sum of the measures of $\angle$ BAD and $\angle$ BAE?

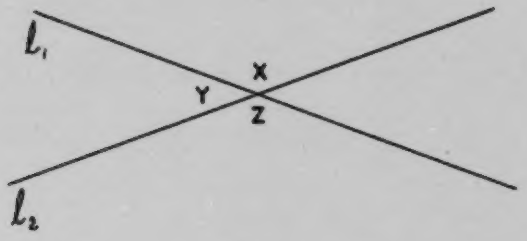
   (b) What is the sum of the measures of $\angle$ CAE and $\angle$ BAE?

   (c) If the measure of $\angle$ BAE is 60, then what is the measure of $\angle$ BAD? Of $\angle$ CAE?

   (d) If the measure of angle BAE is 70, then what is the measure of $\angle$ BAD? Of $\angle$ CAE?

   (e) What can you say about the measures of angles BAD and CAE when the measure of angle BAE changes? Explain why angles BAD and CAE in Figure 9-1-f must have the same measure.

*6. The following figure is similar to Figure 9-1-f. Let x, y, and z represent the angles DAB, BAE, and CAE respectively. The angles are indicated in this way in the figure.



Copy and complete the following statements:

(a) m($\angle$ x) + m($\angle$ y) = ___?___.

(b) m($\angle$ z) + m($\angle$ y) = ___?___.

(c) If m($\angle$ y) is known, how can you find m($\angle$ x)? How can you find m($\angle$ z)?

(d) Write your answer for part (c) in the form of a number sentence as is done in parts (a)and (b) by copying and completing the following:

m($\angle$ x) = ___?___ - ___?___

m($\angle$ z) = ___?___ - ___?___

(e) Write a number sentence to show the relation between m($\angle$ x) and m($\angle$ z).

*7. Imagine two lines in space.

(a) Is there any possible relation between two lines in space which would not occur between two lines in a plane?

(b) Find an illustration in your classroom to explain your answer.

9-2. Three Lines in a Plane

The Greeks were the first to study geometry as a body of knowledge. Thales (640 B.C. - 546 B.C.) studied in Egypt and introduced the study of geometry into Greece. The discovery of the property dealing with pairs of vertical angles which we studied in the previous section is credited to Thales. As you study geometry you will learn about other properties that he and other Greek mathematicians discovered.

In the previous section we primarily studied figures formed by two lines in a plane. In this section we will study figures formed by three lines in a plane.

Draw a figure similar to 9-1-a. Can you draw another line, ℓ_3, through point A? Can lines ℓ_1, ℓ_2, and ℓ_3 have a common point of intersection? Your figures should look something like this:

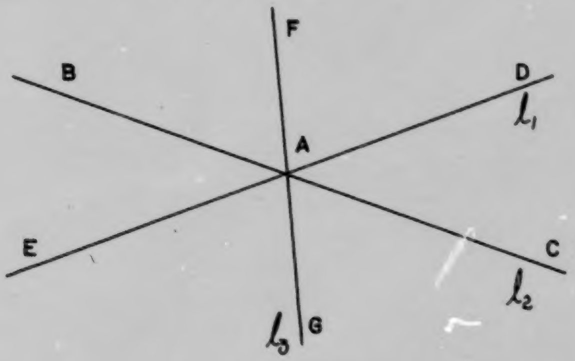


Figure 9-2-a

Will three lines drawn on a plane always have a common point of intersection? Look at Figure 9-2-b where line t intersects lines ℓ_1 and ℓ_2. In the language of sets we would say $\ell_1 \cap t$ is not the empty set, and $\ell_2 \cap t$ is not the empty set. A line which intersects two or more lines in distinct points is called a transversal of those lines. Since line t intersects ℓ_1 and ℓ_2, t is a transversal of lines ℓ_1 and ℓ_2.

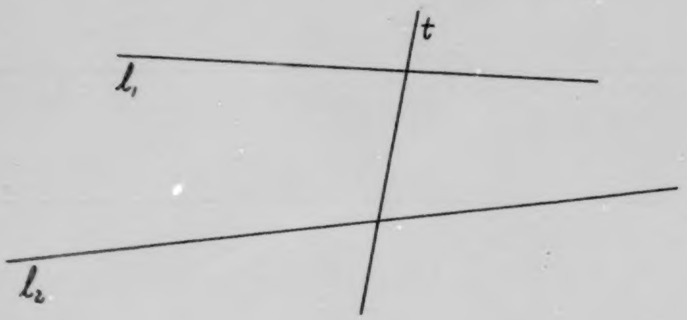
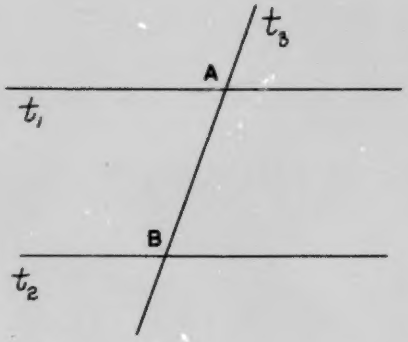


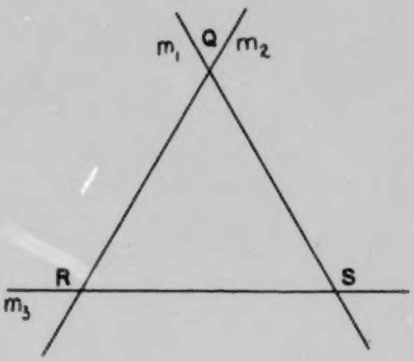
Figure 9-2-b

Exercises 9-2



1. Draw a figure similar to the one at the right. Lines t_1 and t_2 do not intersect. ($t_1 \cap t_2$ is the empty set.) Line t_3 intersects t_1. Call the point of intersection A. Line t_3 intersects t_2. Call the point of intersection B.
   (a) How many pairs of vertical angles are there in your figure?
   (b) How many pairs of adjacent angles are there in your figure?
   (c) Is line t_3 a transversal of lines t_1 and t_2?
2. Draw two lines, m_1 and m_2, which intersect in point Q. Draw a third line, m_3, which intersects m_1 and m_2, but not in point Q. Call the intersection of m_1 and m_3 point S, and call the

intersection of m_2 and m_3 point R.



(a) What is the name of the set of points made up of segments $\overline{SQ}$, $\overline{QR}$, and $\overline{RS}$ in your figure? (See Chapter 4 if you don't know.)

(b) How many pairs of vertical angles are there in your figure?

(c) How many pairs of adjacent angles are there in your figure?

3. Use Figure 9-2-c in answering the following questions:

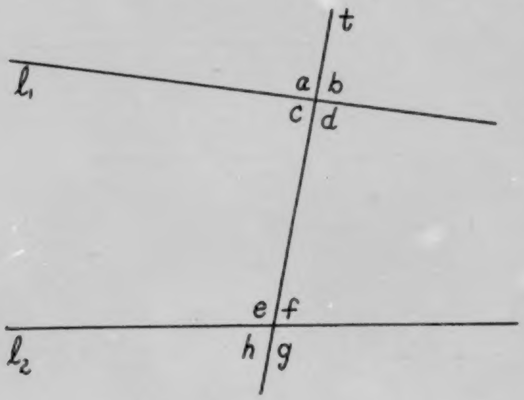


Figure 9-2-c

(a) Name the transversal and tell what lines it intersects.

(b) In this drawing, how many angles are determined by the three lines?

(c) How many pairs of vertical angles are there in this drawing?

(d) What do you know about the measures of each of a pair of vertical angles?

(e) Does m(angle h) = m(angle f)? How can you tell?

(f) Does m(angle c) = m(angle d)? How can you tell?

4. (a) How many pairs of adjacent angles are there in Figure 9-2-c?

(b) What do you know about the sum of the measures of any pair of adjacent angles in Figure 9-2-c?

5. (a) Are angles c and d in Figure 9-2-c supplementary?

   (b) Are angles a and d in Figure 9-2-c supplementary?

   (c) Are the angles in each pair of adjacent angles in Figure 9-2-c supplementary?

   (d) If the measure of $\angle$ h is 80, what is the measure of $\angle$ g? Of $\angle$ e? Of $\angle$ f?

   (e) If the measure of $\angle$ h is 90, are angles h and f supplementary?

   (f) If m($\angle$ h) = m($\angle$ g), are angles e and g supplementary?

6. Notice that one of the rays which forms angle b in Figure 9-2-c is a part of a ray which forms angle f, and the interiors of both the angles are on the same side of the transversal. Angles placed in this way are called corresponding angles.

   (a) Name another pair of corresponding angles on the same side of the transversal as angles b and f in Figure 9-2-c.

   (b) Are angles a and e corresponding angles? (Note that the ray on line t which forms angle a is only a part of the ray on line t which forms angle e, but none of the rays forming angle e are parts of the rays forming angle a.)

   (c) Are $\angle$ c and $\angle$ g corresponding angles?

   (d) How many pairs of corresponding angles are in Figure 9-2-c?

   (e) If the measure of $\angle$ b is 80, can you tell what the measure of $\angle$ f is?

(f) If the measures of $\angle a$ and $\angle e$ are 90, what can you say about the measures of all the angles in Figure 9-2-c?

(g) If the measures of $\angle a$ and $\angle e$ are 90, are angles h and b supplementary angles?

*7. (a) How are the figures in Problems 1, 2, and 3 alike?

(b) How are the figures different?

(c) Can you think of another way to draw a set of three lines in a plane? (Do not use the intersection of three lines in a point.)

(d) Copy and complete the following statement:

"The union of the intersections of each two lines of a group of three different lines in the same plane consists of __?__, __?__, __?__, or __?__ points."

*8. BRAINBUSTER. Look around you and find illustrations of sets of three lines like those in the figures you drew in this exercise. Now imagine the intersection of three planes in space. What figures are formed?

9-3. Parallel Lines and Corresponding Angles

A civil engineer must build two straight roads intersecting a highway, but not intersecting each other. In the following figure, the highway is represented by the transversal, t , and the roads are represented by the lines r_1 and r_2.

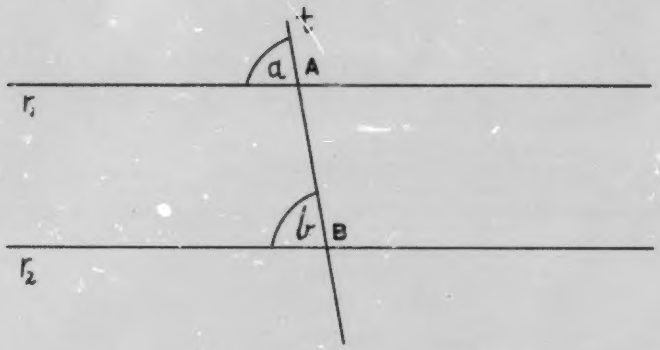


Figure 9-3-a

In the language of geometry, the engineer is asked to build the roads so that

$r_1 \cap t$ is not the empty set,

$r_2 \cap t$ is not the empty set,

$r_1 \cap r_2$ is the empty set.

That is, the engineer wants to build the roads so that r_1 and r_2 will be parallel. Neither r_1 nor r_2 is parallel to t.

The engineer wants to tell his workers what angles the road should form with the highway.

Class Exercise and Discussion

1. Make a drawing like that above, with points A and B about $1\frac{1}{2}$ inches apart. Make the measure of $\angle a$ 70 and the measure of $\angle b$ 40. Do the roads intersect if extended? If so, on which side of t, the left or the right?
2. Draw another figure making the measure of $\angle a$ 30 and that of $\angle b$ 40. Do the roads intersect if extended? If so, on which side of t, the left or the right?

3. Make at least six experiments of this kind with various measures for angles a and b. In at least two cases use the same measure for ∠ a that you use for ∠ b. Record your results like this:

Measure of ∠ a in degrees	Measure of ∠ b in degrees	Intersection of r_1 and r_2
70	40	left of t
30	40	right of t
40	40	?

4. Copy the following table. Predict whether r_1 and r_2 intersect and, if so, where. Make a drawing to check your prediction. (You may extend the table and choose your own measures of angles if you wish.)

Measure of ∠ a in degrees	Measure of ∠ b in degrees	Intersection of r_1 and r_2
50	80	
50	50	
50	40	

5. After class discussion of Problems 1 through 4, copy and complete the following statement:

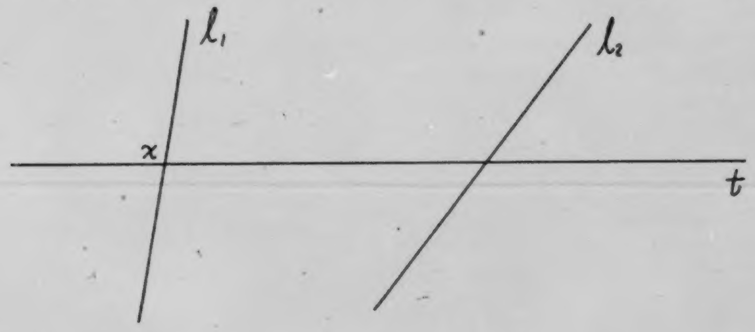
Property 2. When a transversal intersects two lines which lie in the same plane and a pair of corresponding angles have different ___?___, then the lines ___?___.

6. What about the case where the corresponding angles have equal measures? Write a statement for this case. Copy and complete:

   Property 2a. When a transversal intersects two lines in the same ? and a pair of corresponding angles have equal measures, then the lines are ? .

Exercises 9-3

1. Make a figure like the one below. What angle forms with angle x a pair of corresponding angles? Label it angle p. If angles x and p have the following measurements, do ℓ_1 and ℓ_2 intersect above t, below t, or are they parallel?



Copy and complete:

	Measure of x in degrees	Measure of p in degrees	ℓ_1 and ℓ_2		
			Are Parallel	Intersect above t	Intersect below t
(a)	120	140			
(b)	120	120			
(c)	120	90			
(d)	90	120			
(e)	90	90			

(f)	90	70			
(g)	40	90			
(h)	40	40			
(i)	40	20			

2. List several examples of parallel lines that you find in the classroom.

*3. Lay a yardstick or a ruler across some of the parallel lines you find in Problem 2 so as to form a transversal. Do this so that the intersections with the parallel lines are at a minimum distance apart. Measure this distance in each case. What is the measure of the angle formed by the transversal and each line when the minimum distance is found?

9-4. Converse (Turning a Statement Around)

We have seen that if certain things are true, then certain other things are true. For example:

(a) "If two angles are vertical angles, then the angles have the same measure."

Suppose we make a new statement by interchanging the "if" part and the "then" part. This is the new statement:

(b) "If two angles have the same measure, then the angles are vertical angles." A statement obtained by interchanging the "if" clause with the "then" clause is called a "converse" of the first statement. The second statement written above is a converse of the first statement.

If you "turn a true statement around," will the new statement always be true? Let us look at two statements and their converses and decide.

1. "If Mary and Sue are sisters, then Mary and Sue are girls."

Converse of 1: "If Mary and Sue are girls, then Mary and Sue are sisters." Is the original statement true? Is the converse also true?

Now consider the next statement:

2. "If Lief is the son of Eric, then Eric is the father of Lief."

Converse of 2: "If Eric is the father of Lief, then Lief is the son of Eric." Is the original statement true? Is the converse true?

We can see from these two illustrations that, if a statement is true, a converse, obtained by interchanging the "if" part and the "then" part, may be true or may be false.

3. Is statement (a) above, dealing with vertical angles, true? Is the converse statement, (b), true? We cannot accept a converse statement as always being true. Sometimes when a true statement is "turned around," a converse is true. Sometimes when a true statement is "turned around," a converse is false.

Exercises 9-4

1. Make a drawing where a converse of statement (a) in Section 9-4 is not true. Must any two angles which have the same measure

always be vertical angles?

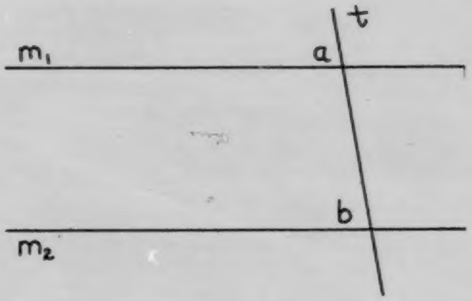
2. For each of the following statements write "true" if the statement is always true; "false" if the statement is sometimes false.
   (a) If Blackie is a dog, then Blackie is a cocker spaniel.
   (b) If it is night, then we cannot see the sun.
   (c) If it is July 4th, then it is a holiday in the United States.
   (d) If Robert is the tallest boy in his school, Robert is the tallest boy in his class.
   (e) If an animal is a horse, the animal has four legs.
   (f) If an animal is a bear, the animal has thick fur.
3. Write a converse for each statement in Problem 2 and tell whether the converse is true of false.
4. Read the following statements. Write "true" if the statement is always true; "false" if the statement is sometimes false.
   (a) If-a figure is a circle, then the figure is a simple, closed curve.
   (b) If a figure is a simple closed curve composed of three line segments, then the figure is a triangle.
   (c) If two angles have equal measures, they are right angles.
   (d) If two lines are parallel, then the lines have no point in common.
   (e) If two angles are supplementary, they are adjacent.

(f) If two adjacent angles are both right angles, they are supplementary.

5. (a) Write and state truth or falsity of a converse statement for Property 2a (Section 9-3):

Property 2a. When a transversal intersects two lines in the same plane and a pair of corresponding angles have equal measures, then the lines are parallel.

(b) To test the truth or falsity of this converse, first draw two parallel lines. This can easily be done by drawing a figure as shown, making corresponding angles a and b of equal measures.



By Property 2a, m_1 and m_2 are parallel. Now draw any other transversal of m_1 and m_2. Call it t_1. Measure the angles in each pair of corresponding angles along t_1. Are they equal? Compare your results with those of your classmates.

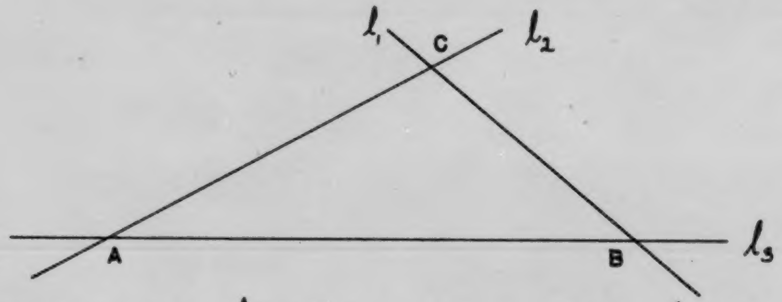
(c) On the basis of this work, do you think the converse of Property 2a stated in (a) above is true or false?

*6. Write and state truth or falsity of a converse statement for Property 2: When a transversal intersects two lines which lie in the same plane and a pair of corresponding angles have different measures, then the lines intersect. Does the statement seem to be true or false? You can test your results by making drawings as in Problem 5.

*7. Can a converse for a false statement be true? If so, can you give an example?

9-5. Triangles

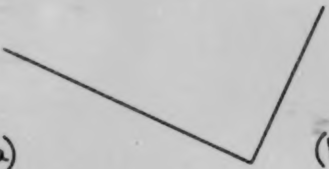
You have been discovering angle relations in a figure composed of three lines, two parallel lines and a transversal. Suppose the lines are arranged so that each line is a transversal for the other two lines. Would such a figure appear like this figure?



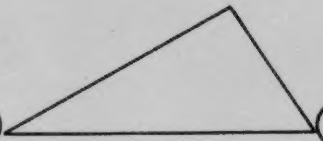
In the drawing above ℓ_1 is a transversal for ℓ_2 and ℓ_3; ℓ_2 is a transversal for ℓ_1 and ℓ_3; ℓ_3 is a transversal for ℓ_1 and ℓ_2. A, B, and C are three points and $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are segments joining them in pairs. In Chapter 4 (Non-Metric Geometry) the union of three points not on the same line and the segments joining them in pairs was said to be a triangle. Our sketch, according to this definition, contains triangle ABC. The points A, B, and C are called the vertices of the triangle and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are called the sides of the triangle.

Which of the figures below are triangles? If a figure is not a triangle, tell which requirement of the definition is lacking.

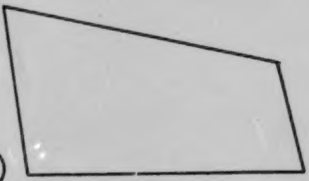
(a)



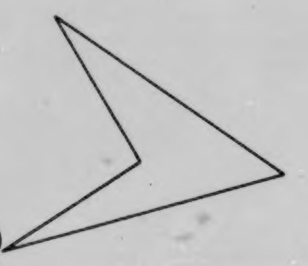
(b)



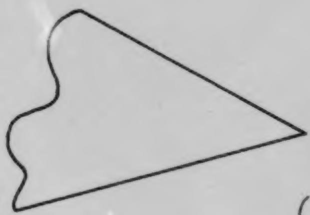
(c)



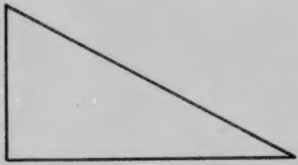
(d)



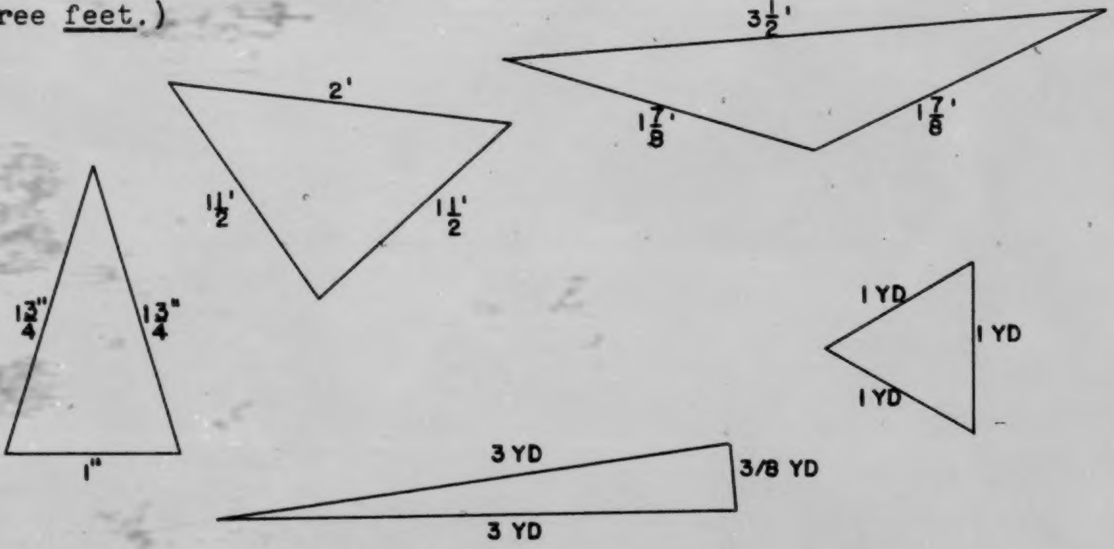
(e)



(f)

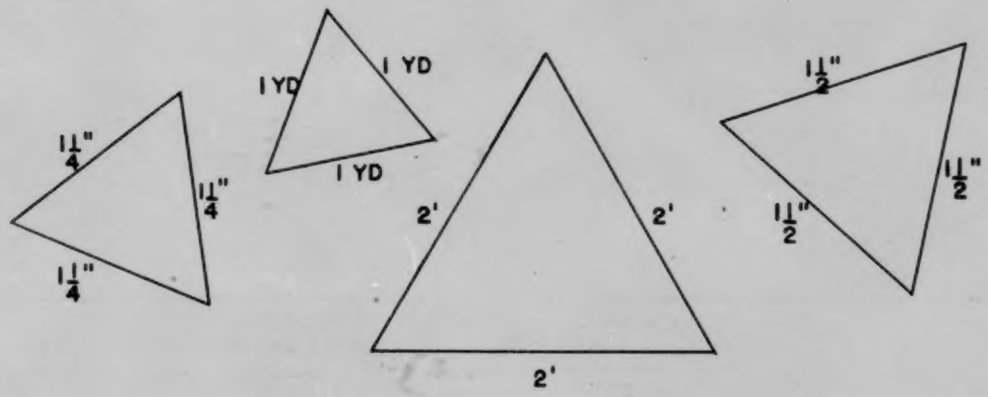


The triangles in the group below are members of a particular set of triangles because they have a common property. What is the common property? What do you notice about the lengths of the sides of the triangles? (The symbol 3" means three inches. The symbol 3' means three feet.)



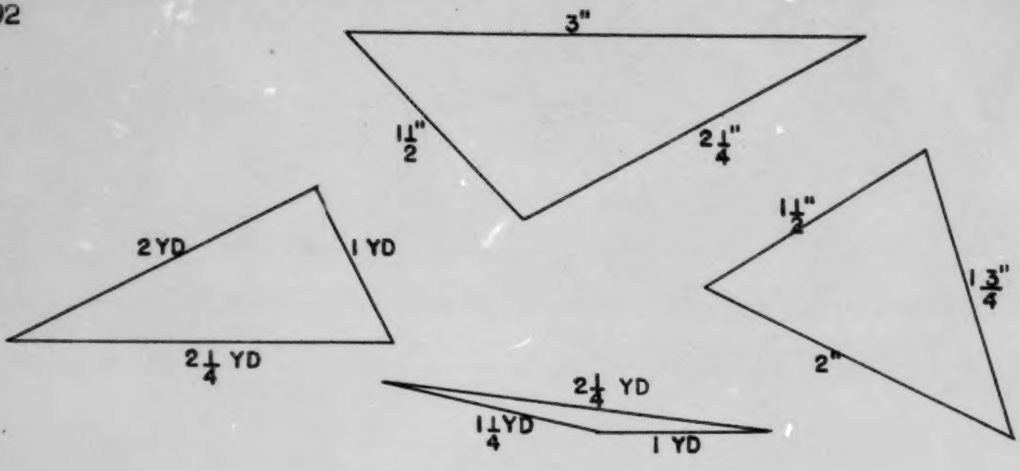
Note that each triangle has at least two sides which are equal in length. The triangles of this set, and all other triangles that share this property, are called isosceles triangles.

The triangles in the group below are members of another particular set of triangles. What is the common property in this set of triangles?

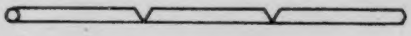


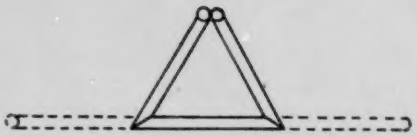
Note that each triangle has three sides which are equal in length. Another way of saying this is that all sides in the same triangle have equal measures. All triangles of this set, and all other triangles that share this property, are called equilateral triangles. The word "equilateral" comes from the words, "equi" meaning equal and "lateral" meaning side.

The triangles in the following group are members of still another particular set of triangles. What is the common property in this set of triangles?

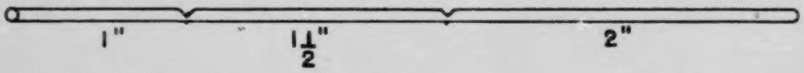

3"
1 1/2"
2 1/4"
2 YD
1 YD
2 1/4 YD
1 1/2"
1 3/4"
2"
2 1/4 YD
1 1/4 YD
1 YD

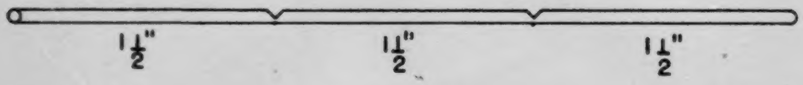
Do two sides in any of the above triangles have the same measures? Do any of the triangles have three sides with equal measures? No two sides in any of the triangles above have the same measure. The triangles in this particular set, and all other triangles having the same property, are called scalene triangles.

Here is a picture of a soda straw and here it is creased at two points  ready to be folded so that the ends come together like this:

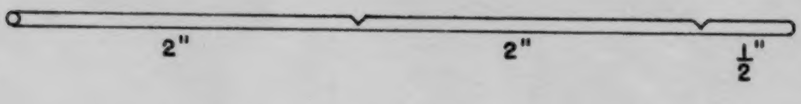


What kinds of triangles could be represented in this way by folding the following straws:

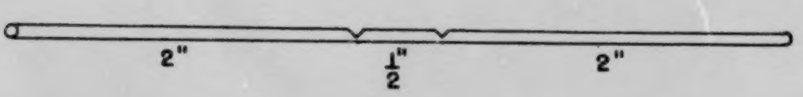
(a) 

(b) 

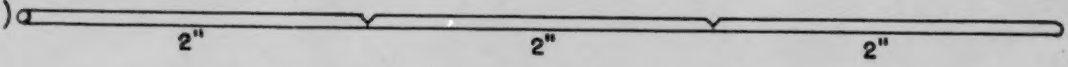
(c)



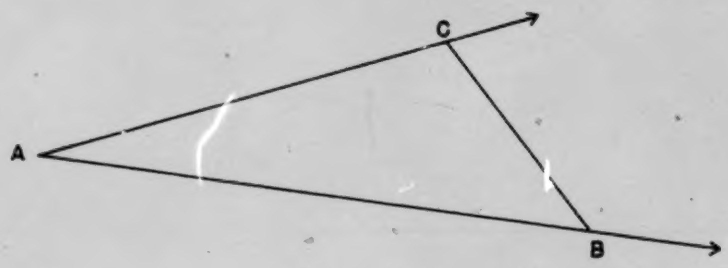
(d)



(e)

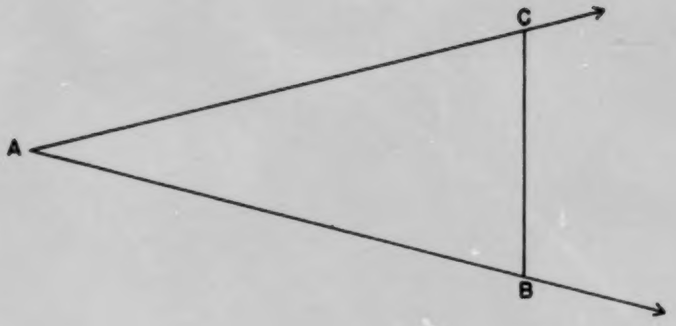


Now you are to draw some triangles. Draw an angle, and name the vertex A. Locate a point B on one ray of the angle, and a point C on the other ray. Draw segment $\overline{BC}$. Your drawing should look something like this.



The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called "triangle ABC." Notice that in your drawing, point C and B are the only points shared by angle A and side $\overline{BC}$. Angle A and side $\overline{BC}$ are said to be opposite each other because their intersection consists of just the endpoints of $\overline{BC}$. Is $\angle B$ opposite $\overline{AC}$? Why? Is $\angle C$ opposite $\overline{AC}$? Why? What other angle and side are opposite each other in your drawing? Why?

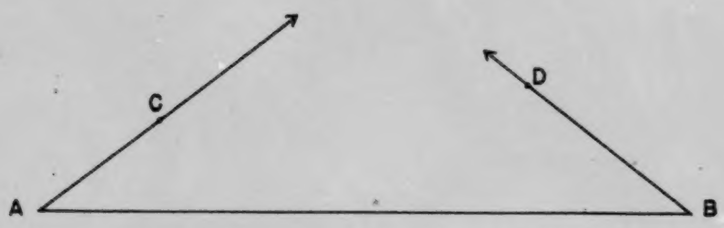
Draw another triangle as before, but this time very carefully locate point B and point C so that $\overline{AB}$ and $\overline{AC}$ are equal in length as shown below:



What kind of a triangle is △ ABC? Why? Make a tracing of angle B and try to fit the tracing upon angle C. What does this procedure suggest about the measure of angles B and C? Make several more tracings and fittings for the isosceles triangles on the second page of this section. What do you observe? Copy and complete the following statement:

Property 3. If two sides of a triangle are __?__ in length, then the angles ____?____ these sides have __?__ measures.

Now make a drawing like the one below. Draw AB about four inches in length. Draw 40 degree angles at A and B as shown. Locate points C and D at some convenient place on the rays of these angles.



Notice that we have drawn two angles whose measures are the same. Extend $\overline{AC}$ and $\overline{BD}$. Name their point of intersection E. Thus triangle ABE is determined.

What do you observe about the sides opposite $\angle A$ and $\angle B$ in this triangle? Make several other sketches according to the directions given. Keep angles A and B acute and equal in measure, but vary the measure for different triangles. What do you observe about the sides in each figure? To what special set of triangles do your figures appear to belong?

Copy and complete the following statement:

Property. If two angles of a triangle are ___?___ in measure, then the sides ___?___ these angles are ___?___ in length.

Have you noticed that this statement resembles the statement completed as Property 3 previously? What new word learned by studying Section 9-4 could be used to describe the relationship between these two statements? We will call this statement the converse of Property 3.

Class Discussion Problems

1. Could the three sets of triangles we have discussed (isosceles, equilateral, and scalene) be determined by some common·property pertaining to angle measures instead of side measures? What advantages or disadvantages would this method of classification have?
2. How could the converse of Property 3 be used to show that a triangle having all three of its angles equal in measure must necessarily be an equilateral triangle?

Exercises 9-5

1. Draw a triangle with two sides of the same measure. Such a triangle is called an isosceles triangle.

2. Draw a triangle which has three sides with the same measure. It is called an equilateral triangle.

3. Draw a triangle in which no two sides have the same measure. Such a triangle is called a scalene triangle.

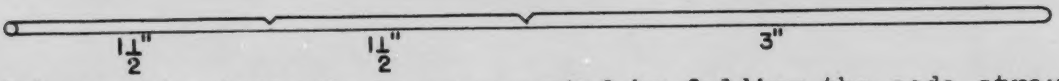
4. If a triangle is isosceles is it also equilateral? Explain your answer.

5. If a triangle is equilateral, is it also isosceles? Explain your answer.

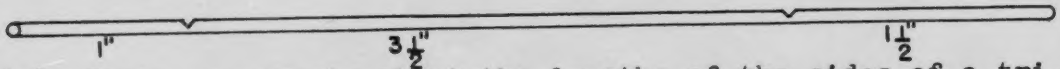
6. Draw an equilateral triangle and letter the vertices P, Q, and R. Match each angle with its opposite side by listing them in pairs.

*7. BRAINBUSTER:

(a) Could a triangle be represented by folding the soda straw shown in this figure? Explain your answer.

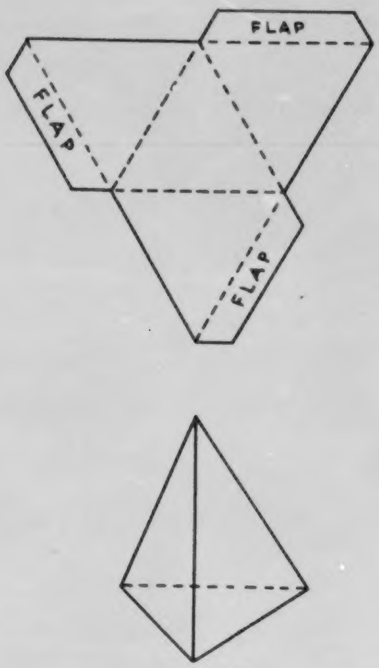


(b) Could a triangle be represented by folding the soda straw shown here? Explain your answer.



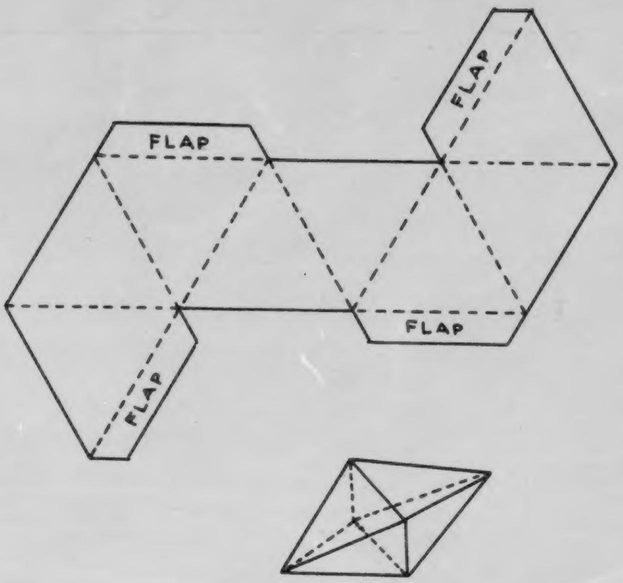
(c) State a property about the lengths of the sides of a triangle as suggested by your observations in parts (a) and (b).

8. OPTIONAL: Copy the following patterns. Fold on the dotted lines after cutting along the edges and paste the flaps. Note the equilateral triangles. Try to enlarge these patterns for better results.



tetrahedron

(four faces)



octahedron

(eight faces)

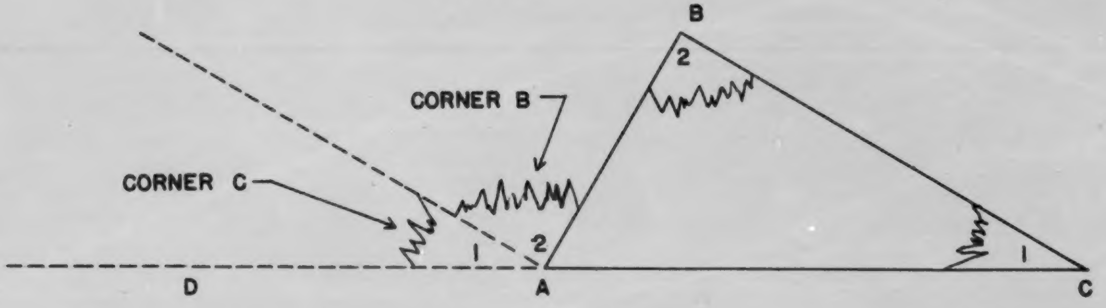
9-6. Angles of a Triangle

In the previous section we studied about special properties for certain triangles. In this section we come to a property which holds for all triangles.

Class Exercises and Discussion

1. Draw a triangle, making each side about two or three inches in length. Cut out the triangular region by cutting along the

sides of the triangle. Tear off two of the corners of this region and mount the whole figure on cardboard or a sheet of paper as shown in the figure below. Note: Corners B and C are placed around the vertex A. (The corners may be pasted or stapled in place.)

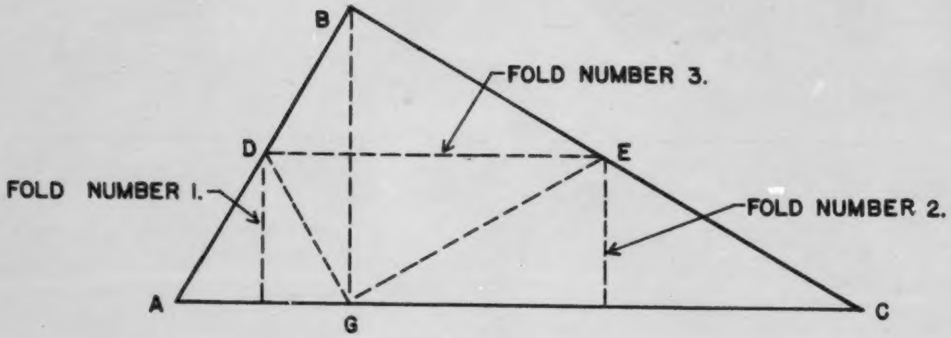


(a) Measure the angle DAC. Compare the results with those of your classmates. What is the measure of $\angle$ DAC?

(b) What do you observe about angle 1, angle 2, and angle BAC in this new arrangement?

2. Cut the interior of a triangle ABC out of paper and mark off the midpoints D and E of the sides $\overline{AB}$ and $\overline{BC}$ respectively. (The midpoint is halfway from one endpoint of a line segment to the other endpoint. Each segment of a pair of segments will then have the same length. Thus $\overline{AD}$ and $\overline{DB}$ will have the same length, and $\overline{BE}$ and $\overline{EC}$ will have the same length.

Fold over the segment $\overline{AD}$ so that A lies over some point G of the line segment AC as shown in the following drawing.



Similarly, fold over $\overline{EC}$. Where does point C seem to fall? Finally, fold along the segment $\overline{DE}$. Where does point B seem to fall?

(a) Where do the segments $\overline{DB}$ and $\overline{EB}$ lie?

(b) What is the sum of the measures of the angles of triangle ABC?

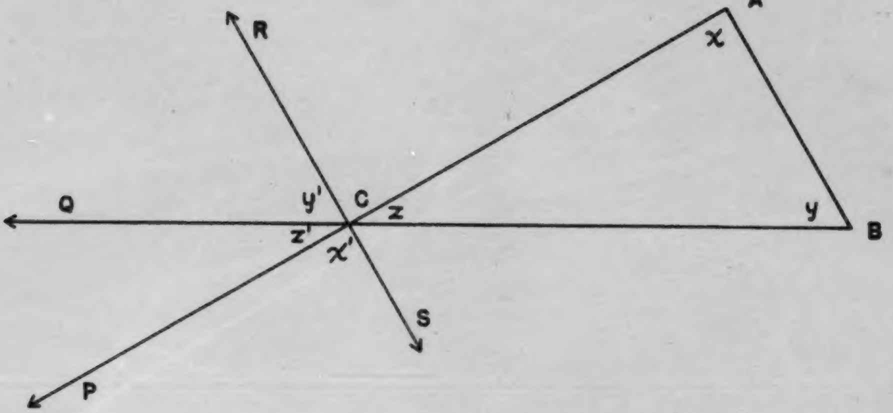
(c) Does this experiment work with other triangles? Check your results with those of your classmates.

(d) Does the property you found in Problem 1 fit these observations?

3. Copy and complete the following statement:

Property 4. The sum of the measures in degrees of the angles of any triangle is ___?____.

4. Consider the triangle ABC shown on the following page with rays $\overrightarrow{AP}$ and $\overrightarrow{BQ}$. Line RS is drawn through point C so that the measure of $\angle y$ and the measure $\angle y'$ are equal. (We will use a new notation, y', to name an angle. y' is read "y prime.")



Use a property to explain "why" in the following questions whenever you can.

(a) Is $\overleftrightarrow{RS}$ parallel to $\overline{AB}$? Why?

(b) What kind of angles are the pair of angles marked x and x' ? Is $m(\angle x) = m(\angle x')$? Why?

(c) What kind of angles are the pair of angles marked z and z' ? Is $m(\angle z) = m(\angle z')$? Why?

(d) $m(\angle y) = m(\angle y')$ Why?

(e) $m(\angle x) + m(\angle y) + m(\angle z) = m(\angle x') + m(\angle y') + m(\angle z')$ Why?

(f) $m(\angle x) + m(\angle y) + m(\angle z)$ is the sum of the measures of the angles of the triangle. Why?

(g) $m(\angle x') + m(\angle y') + m(\angle z') = 180$ Why?

(h) $m(\angle x) + m(\angle y) + m(\angle z) = 180$ Why?

(i) We conclude therefore that the sum of the measures of the angles of the triangle is 180. Why?

This is a proof of Property 4. We will not study many proofs this year, but in some Exercises you may be asked to try to discover a proof. As you study more geometry in later years you should become quite able to discover proofs.

Notice that in this proof we drew just any triangle. Does the proof apply to all triangles? If you are in doubt about this, you might draw some other triangles quite different in shape from the one in this section, label points, angles, segments, rays, and lines in the same way. Then, try the proof above for the figure you have drawn.

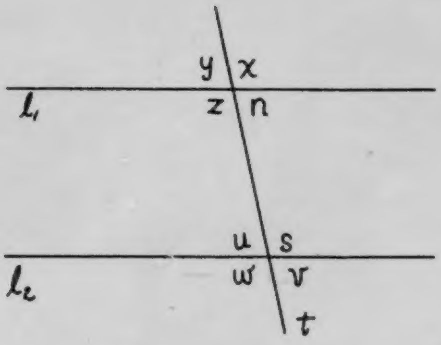
Exercises 9-6

1. What is the measure of each angle of an equilateral triangle?
2. What is the measure of the third angle of the triangles if two of the angles of the triangle have the following measures?
   (a) 40 and 80.
   (b) 100 and 50.
   (c) 70 and 105.
   (d) 80 and 80.
3. Suppose one angle of an isosceles triangle has a measure of 50. Find the measures of the other two angles. Are two different sets of answers possible?
4. If two triangles, ABC and DEF, are drawn so that m(BAC) = m($\angle$ EDF) = m($\angle$ BCA) = m($\angle$ EFD), what will be true about angles ABC and DEF? Upon what property is your answer based?

5. In each of the following the measures of certain parts of the triangle ABC are given, those of the sides in inches and those of the angles in degrees. You are asked to predict the measure of some other part. In each case, give your reason.

Given	Find	Property
(a) $m(\angle ABC) = 60$, $m(\angle BCA) = 40$.	$m(\angle CAB)$	?
(b) $m(\angle CAB) = 52$, $m(\angle BCA) = 37$.	$m(\angle ABC)$	?
(c) $m(\angle ABC) = 40$, $m(\overline{AB}) = 2$, and $m(\overline{AC}) = 2$.	$m(\angle ACB)$	?
(d) $m(\overline{AB}) = 3$, $m(\overline{AC}) = 3$, $m(\overline{BC}) = 3$.	$m(\angle BCA)$	?
(e) $m(\angle BAC) = 100$, $m(\angle BCA) = 40$, and $m(\overline{AB}) = 4$.	$m(AC)$	?

*6. In the figure at the right ℓ_1 and ℓ_2 are parallel.



(a) Does $m(\angle y) = m(\angle n)$? Why?

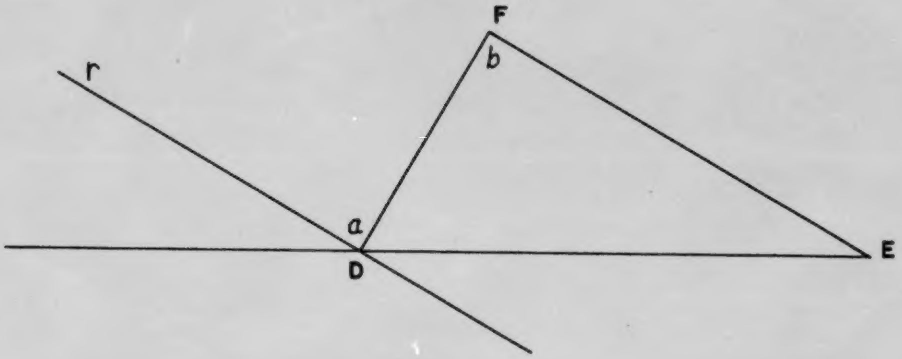
(b) Does $m(\angle y) = m(\angle u)$? Why?

(c) Try to prove that $m(\angle n) = m(\angle u)$.

(d) Discuss with your classmates the things which make a proof good.

(e) After the class discussion, rewrite your proof in accordance with the points brought out by you and your classmates.

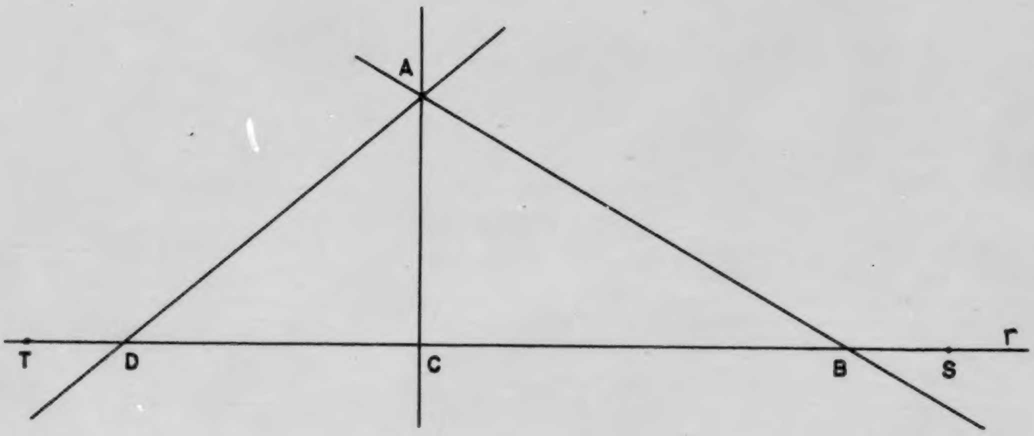
*7. By the proof asked for in Problem 6, it has been proved that the measures of angles a and b in the figure below are equal if line r is parallel to $\overleftrightarrow{BC}$. Use this property to prove that the sum of the measures of the triangle DEF shown below is 180.



9-7. Parallelograms

Distance Between Parallel Lines

Look at the figure below showing lines through point A and meeting line r.



Use a protractor to check the measures of angles given below. The measurement of angles here is in degrees.

$m(\angle SDA) = 40$ $m(\angle SCA) = 90$ $m(\angle SBA) = 150$

Give the following measures.

$m(\angle TDA) = ?$ $m(\angle TCA) = ?$ $m(\angle TBA) = ?$

Measure the segments $\overline{AD}$, $\overline{AC}$, and $\overline{AB}$.

Which is the shortest?

Copy the figure and draw other lines through A which intersect r. Measure the segments on these lines from A to the intersections of the lines with r.

Do you find any of these segments that are shorter than $\overline{AC}$? Note that $\overleftrightarrow{AC}$ is perpendicular to r.

On the basis of your experience here, copy and complete the following statement:

> The shortest segment from a point A to a point of a line r is the segment from A which is ________ to r.

The length of this shortest segment is often called the distance from A to r.

In the following figure, k_1 and k_2 represent parallel lines. Lines a, b, c, are lines through three points D, E, F of k_1 which are perpendicular to k_2. That is, the lengths of $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$ are the distances from F, E, and D to line k_2. One often draws a small square, as in the figure at A, B, C, to indicate that an angle is intended to be a right angle.

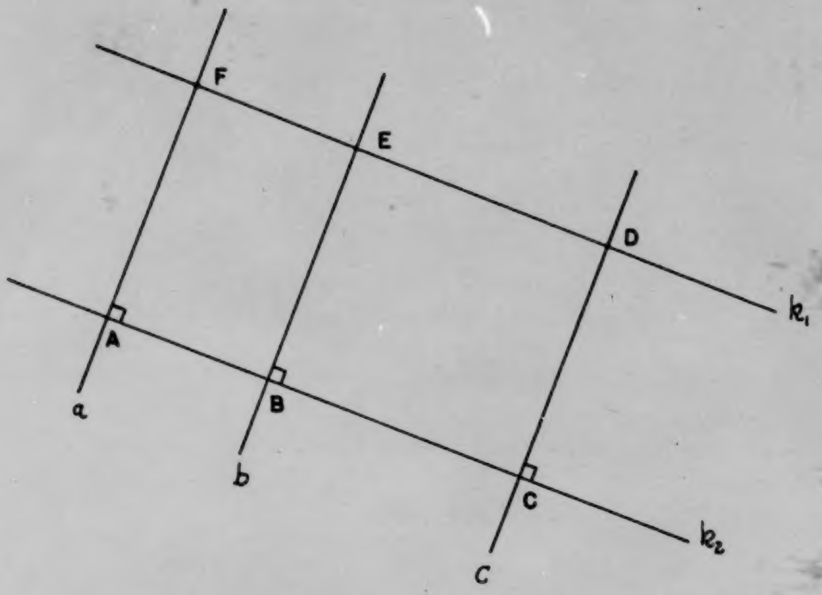
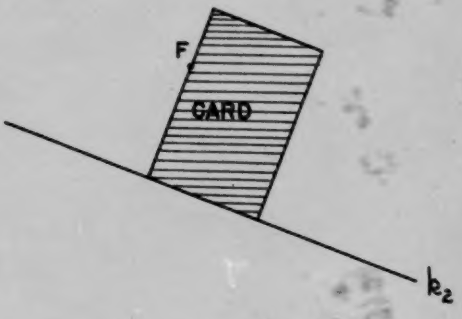


Figure 9-7-a

Drawing a line, such as a through D and perpendicular to k_2, can be conveniently done by using a card or sheet of paper with a square corner. The figure at the right illustrates this.



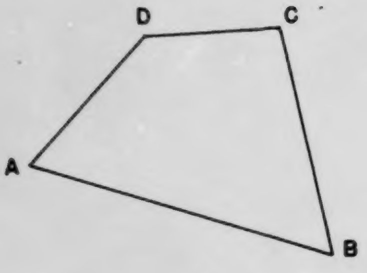
In Figure 9-7-a above there are 21 other right angles besides those marked. Can you find them? How do you know they are all right angles?

In Figure 9-7-a measure the lengths of $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$. Do they appear to be equal in length? This common length is called the distance between lines k_1 and k_2 . Thus the distance between two parallel lines may be described as the length of any segment contained in a line perpendicular to the two lines, and having an endpoint on each of the lines.

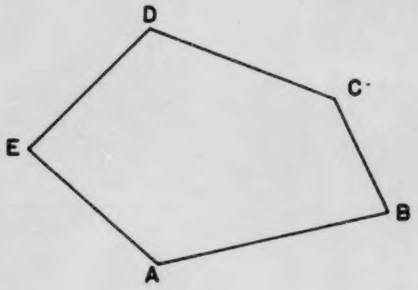
Parallelograms

In Chapter 4 you were introduced to the idea of a simple closed curve. A simple closed curve which is a union of segments is called a polygon. In later work you may sometimes see the word "polygon" applied to curves which are not simple, but for us it will mean a simple closed curve. Unless it is indicated otherwise, we shall also understand that a polygon lies in a plane.

Polygons with different numbers of sides (i.e., segments) are given special names. You already know that a polygon with three sides is called a triangle. Similarly a polygon with four sides is called a quadrilateral, and a polygon with five sides is called a pentagon.



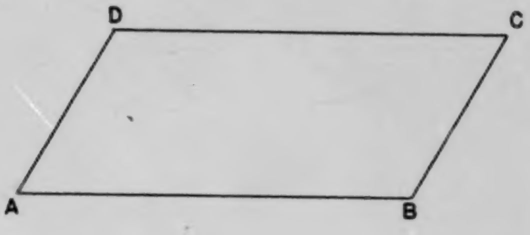
Quadrilateral



Pentagon

In a quadrilateral two sides (segments) which do not intersect are called opposite sides. (How would you describe two opposite sides by referring to their intersection?) Name the pairs of opposite sides in the above quadrilateral.

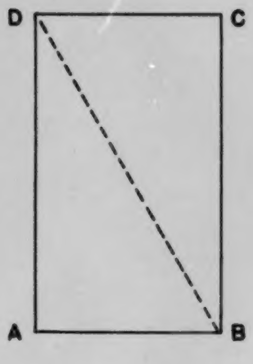
A particularly important kind of quadrilateral is the parallelogram. This is a quadrilateral whose opposite sides lie on parallel lines. The figure ABCD below represents a parallelogram. Name its pairs of opposite sides.



In the future, if two segments lie on parallel lines we will speak of the segments as parallel. Thus we can say that the opposite sides of a parallelogram are parallel.

Property 5. A pair of opposite sides of a parallelogram are parallel and equal in length.

Of course we already know opposite sides are parallel. For the equality,measure the sides in the parallelogram above. Do you find that the opposite sides have equal length? Draw several parallelograms and measure the lengths of the pairs of opposite sides. Do you agree with the property (underlined) above?



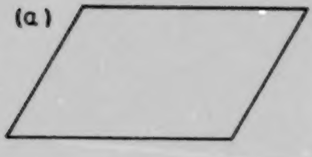
The figure at the right represents a rectangular sheet of paper. (Notice that a rectangle is a special kind of parallelogram.) Take such a piece of paper and tear or cut it into two parts by first folding it along the dotted line as shown. By laying one piece on the other, show that the interiors of triangles ABD and BCD have the same size. Is the area of the interior of one of the triangles equal to one-half the area of the interior of the parallelogram? Why?

In Problem 5 below you are asked to repeat the above steps using parallelograms of different shapes.

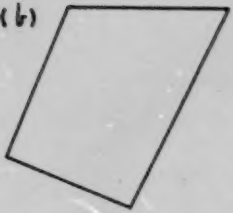
Exercises 9-7

1. Identify several pairs of parallel lines in your classroom, and measure the distances between them.
2. Identify several examples of parallelograms in your classroom.
3. Which of the following figures are parallelograms, assuming that segments which appear to be parallel are parallel?

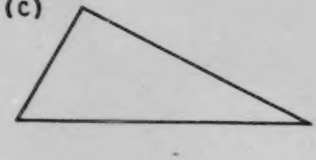
(a)



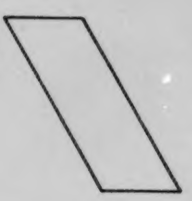
(b)



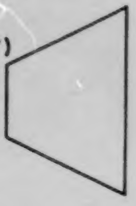
(c)



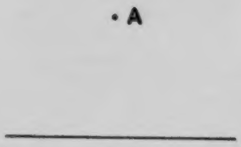
(d)



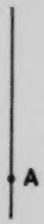
(e)



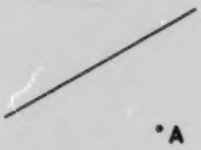
4. Draw perpendiculars to lines in approximately the positions illustrated through points A as indicated. Use a separate piece of paper. Do not draw in your book.



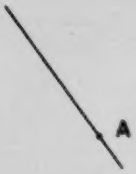
(a)



(b)



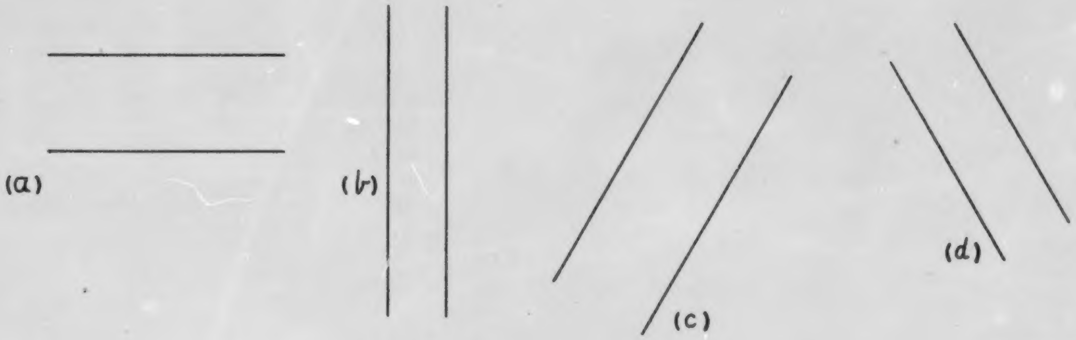
(c)



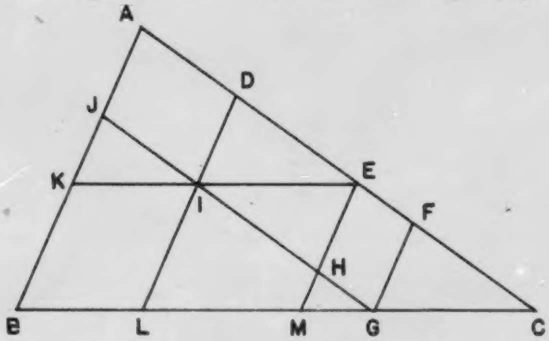
(d)

5. Draw a parallelogram and cut carefully along its sides. The resulting paper represents the interior of the parallelogram. Draw a diagonal (a line joining opposite vertices) and cut the paper along this diagonal. Compare the two triangular pieces. What do you conclude about these triangular pieces? Carry out the same process for two other parallelograms of different shapes. Write a statement that appears to be true on the basis of your experience in the problem.

6. For each of the sets of parallel lines in the figure below, draw a line perpendicular to one of them. (Do not write in your book. Copy the lines in approximately these positions on a separate piece of paper.) Are the lines which are perpendicular to one of two parallel lines perpendicular to the other also?



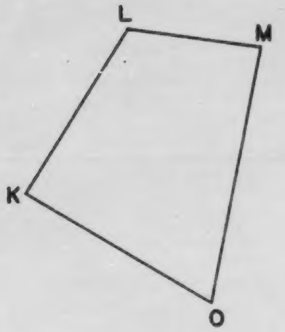
7. List the parallelograms in the following figure.



Assume the segments $\overline{AB}$, $\overline{DL}$, $\overline{EM}$, $\overline{FG}$ are parallel, that the segments $\overline{GJ}$ and $\overline{AC}$ are parallel, and that segments $\overline{KE}$ and $\overline{BC}$ are parallel. (There are 9 such parallelograms.)

8. Without measuring, list pairs of segments in the above figure which are equal in length. Should there be 18 such pairs because there are 9 parallelograms, or are there possibly more?

9. The following questions refer to a figure which is a quadrilateral, as suggested by the drawing. Each question, however, involves a different quadrilateral.



(a) $\overline{KL}$ is parallel to $\overline{OM}$, $\overline{LM}$ is parallel to $\overline{KO}$, $\overline{KL}$ has a length of 3 in. and $\overline{OK}$ a length of 6 in. What are the lengths of $\overline{LM}$ and $\overline{OM}$?

(b) $\overleftrightarrow{KL} \cap \overleftrightarrow{OM}$ is the empty set, $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set. $\overline{LM}$ has a length of 4 in. and $\overline{OM}$ is three times as long as $\overline{LM}$. Find the lengths of $\overline{KL}$ and $\overline{OK}$.

(c) $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set, $\overleftrightarrow{OM} \cap \overleftrightarrow{KL}$ is not the empty set. Can two opposite sides have the same length? Can both pairs of opposite sides have this property? (Draw figures to illustrate this.)

10. Copy and complete the following statement of a property: If a line is perpendicular to one of several parallel lines, then it is ________________________________.

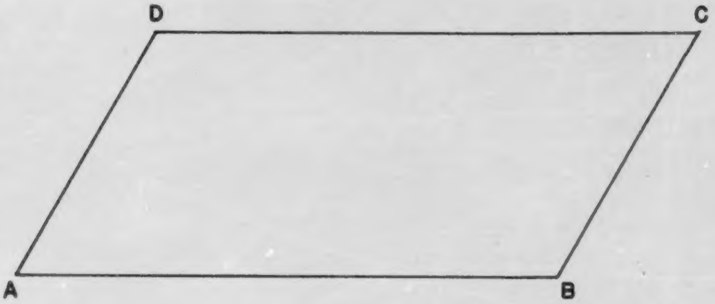
Why is this property true?

*Prove the property. (You may assume that a line perpendicular to one of two parallel lines does intersect the other.)

9-8. Areas of Parallelograms and Triangles

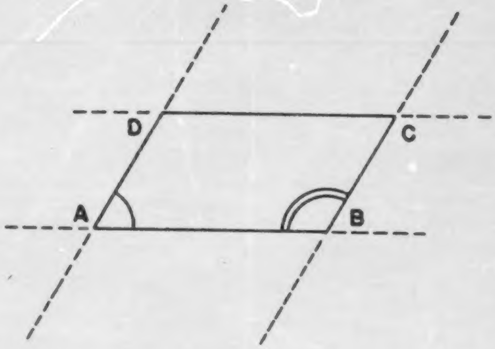
A parallelogram, as you have learned, is a quadrilateral (a four sided figure) in which the opposite sides lie on parallel lines. Does the set of parallelograms contain the set of rectangles?

As you have already discovered, the opposite sides of a parallelogram have the same length. Which pairs of sides in the figure below have equal lengths?



At each vertex of the parallelogram there is an angle whose interior contains the interior of the parallelogram. At vertex A this is the angle BAD. Name the angles at each of the other vertices. These are called the angles of the parallelogram. If the sides of the parallelogram are extended by dotted lines as shown in the figure on the following page, there are several cases of parallel lines cut by a transversal. How many cases do you find?

Copy the figure on the following page and mark the angle of the parallelogram at vertex A as shown. We will call this angle of the parallelogram "angle A." Similarly mark angle B of the parallelogram as shown. Mark all the angles in your figure which have the same measure as angle A in the same way angle A is marked. Do the same for angles having the same measure as angle B. In each case give the reason why the label is correct.



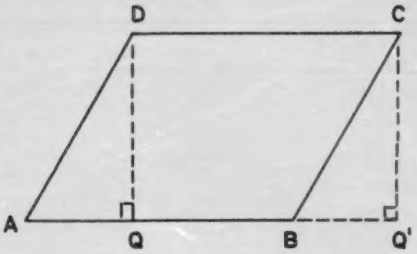
If you have proceeded correctly, all the angles determined by the lines in the figure are now labeled. On the basis of the results just obtained, complete the following statements:

m(∠ A) + m(∠ B) = ___?___

The angles of a parallelogram at two consecutive vertices are ___?___.

The measures of the angles of a parallelogram at two opposite vertices are ___?___.

According to the results above, what can you conclude about the parallelogram if ∠ A is a right angle? Can you be sure it is a rectangle? (Do you recall from Chapter 7 how to find the area of a rectangle?) If the figure is not a rectangle then ∠ A and ∠ B are not right angles, and one of them is an acute angle. Why? Suppose the acute angle is ∠ A. From point D, draw the segment $\overline{DQ}$ perpendicular to the base $\overline{AB}$ of the parallelogram as shown in the figure at the right. Since we know that $\overline{AD}$ and $\overline{BC}$ have the same length, imagine the triangle AQD moved rigidly, that is, without changing its size and shape, into the position of triangle BQ'C. Then point Q' lies on the extension of $\overline{AB}$. How do you know?



The figure QQ'CD is therefore a rectangle. (How do you know it has right angles at C and D?) Moreover the rectangle QQ'CD and the parellelogram ABCD are made up of pieces of the same size and thus have the same area according to Property 3 in Section 2 of Chapter 7. To find the area of the parellelogram it is only necessary therefore to find the area of the rectangle, which we already know how to do. If you do not recall this, refer to Section 7 of Chapter 7.

Notice that the base $\overline{AB}$ of the parallelogram has the same length as side $\overline{QQ'}$ of the rectangle. How do you know that is true? The other side of the rectangle, $\overline{QD}$, is a segment perpendicular to the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Such a line is called an altitude of the parallelogram to the base $\overline{AB}$. The length of the altitude is the distance between the two parallel lines and, as we have already seen, is the same wherever measured. Thus it actually makes no difference whether we consider the altitude from D or that from C or that from any point of $\overline{DC}$.

On the basis of the discussion above copy and complete the following statement: "The number of square units of area in the interior of a parallelogram is the ___?___ of the number of linear units in the _____?_____ and the number of linear units in the ____?____ to this base.

Either side of a parallelogram may be considered as a base. For example, examine the figures below where the same parallelogram is looked at in two ways.

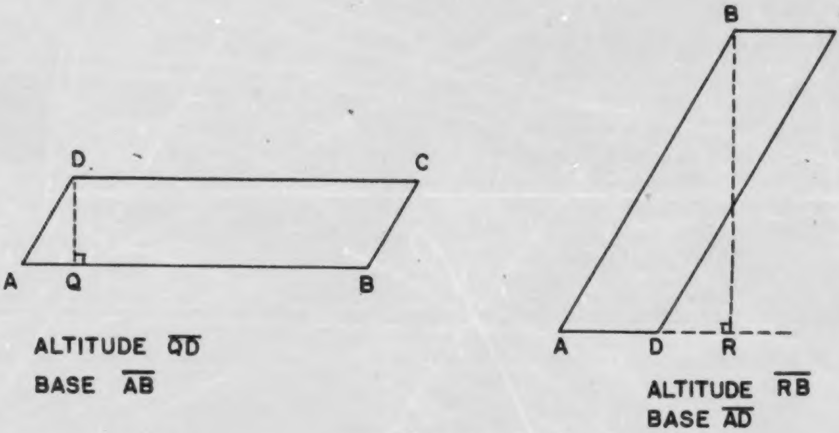
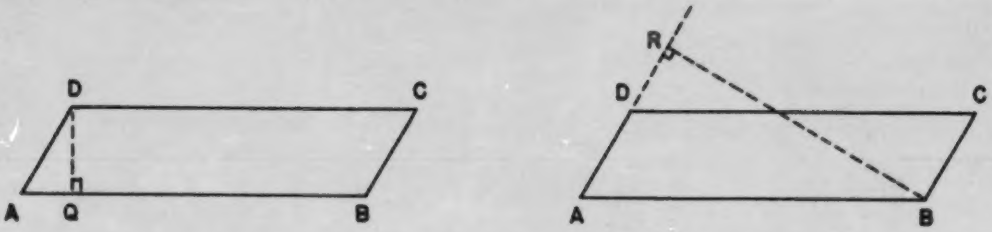

D
C
A
Q
B
ALTITUDE $\overline{QD}$
BASE $\overline{AB}$
B
C
A
D
R
ALTITUDE $\overline{RB}$
BASE $\overline{AD}$

Figure 9-8-a

In Figure 9-8-a we have, for convenience in each case, turned the parallelogram into a position so the base is horizontal. This is not at all necessary, and you should practice thinking of any side of a parallelogram as a base, no matter what its position. For example, it might have been better to show the two cases in Figure 9-8-a without moving the parallelogram, as follows.



BASE $\overline{AB}$
ALTITUDE $\overline{QD}$

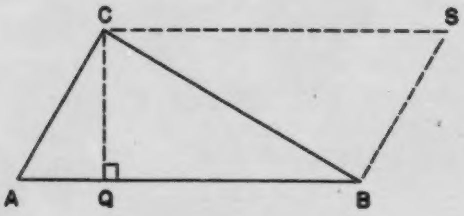
BASE $\overline{AD}$
ALTITUDE $\overline{RB}$

Figure 9-8-b

Thus we really have two ways of finding the area of any parallelogram. The discussion above actually does not cover the case in the right hand figure above where the point R does not lie on the segment $\overline{AD}$. The rule you stated above is true in any case however. For the justification in the case of the figure on the right see the "Brainbusters" below.

Notice that if a parallelogram is a rectangle, the lengths of the base and the altitude are simply the "length" and "width" of the rectangle, so the method above is the same as our earlier one in Chapter 7.

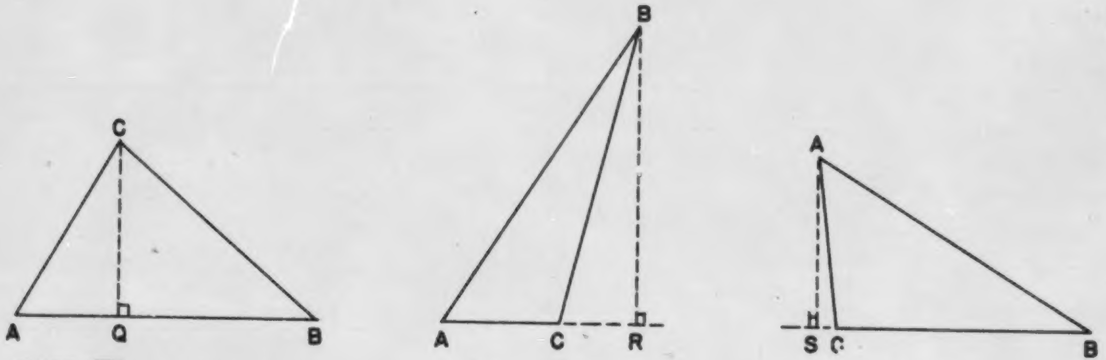
Consider any triangle ABC as shown at the right. Through C and B draw lines parallel to segments $\overline{AB}$ and $\overline{AC}$ and meeting in some point S. The figure ABSC is therefore a parallelogram. The segment $\overline{CQ}$ through C perpendicular to line $\overleftrightarrow{AB}$ is called the altitude of the triangle ABC to the base $\overline{AB}$. The length



of altitude $\overline{CQ}$ is the distance from C to line $\overleftrightarrow{AB}$. Notice that $\overline{AB}$ and $\overline{CQ}$ are also a base and an altitude of the parallelogram.

In Section 7 you discovered that the interiors of triangles ABC and SCB are the same size. Since their interiors cover the whole interior of the parallelogram, it follows that the area of the interior of triangle ABC is one-half that of the interior of the parallelogram ABSC. Using the method above for the parallelogram, copy and complete the following statement: "The number of square units in the area of the interior of a triangle is _______ the ________ of the number of linear units in the ______ and the number of linear units in the _______ to this base."

Since any side of a triangle may be considered as the base, this rule actually gives three ways of finding the area of any triangle. This is illustrated in the following figures showing the same triangle using each of the 3 sides as the base:



BASE $\overline{AB}$
ALTITUDE $\overline{QC}$

BASE $\overline{AC}$
ALTITUDE $\overline{RB}$

BASE $\overline{CB}$
ALTITUDE $\overline{SA}$

Figure 9-8-c

Here the same triangle has been shown in three positions, so that in each case the base is shown as horizontal. This is not necessary, and you will want to practice thinking of the bases and corresponding altitudes in different positions. For example, it might have been better in Figure 9-8-c not to move the triangle but to show the three cases as follows:

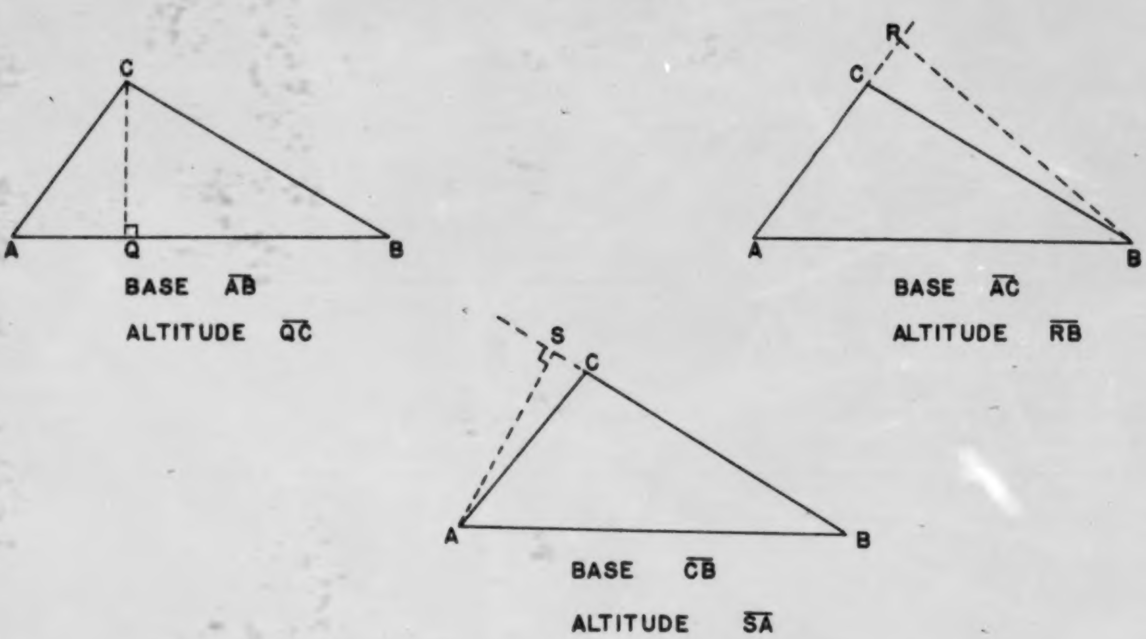
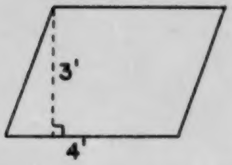

C
A
Q
B
BASE $\overline{AB}$
ALTITUDE $\overline{QC}$
R
C
A
B
BASE $\overline{AC}$
ALTITUDE $\overline{RB}$
S
C
A
B
BASE $\overline{CB}$
ALTITUDE $\overline{SA}$

Figure 9-8-d

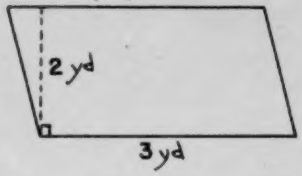
Exercises 9-8

1. Find the areas of the interiors of the parallelograms shown, using the dimensions given.

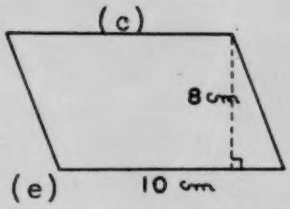
(a)



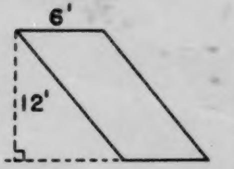
(b)



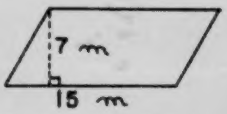
(c)



(d)

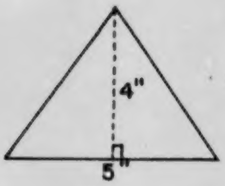


(e)

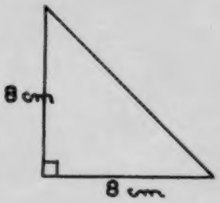


2. Find the areas of the triangles shown using the dimensions given.

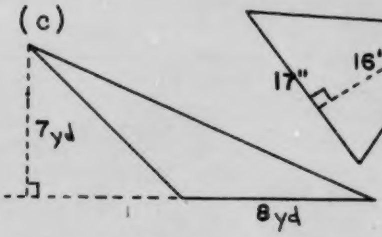
(a)



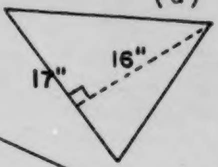
(b)



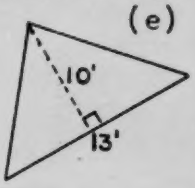
(c)



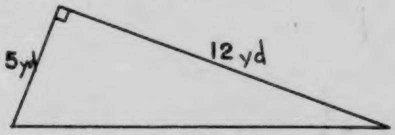
(d)



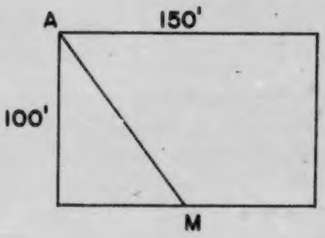
(e)



3. A right triangle has sides of 5 yards and 12 yards as shown. Find the number of square yards in its interior.



4. A man owned a rectangular lot 150 ft. by 100 ft. From one corner, A, there was a path to the point M in the center of the longer opposite side as shown. Find the areas of the

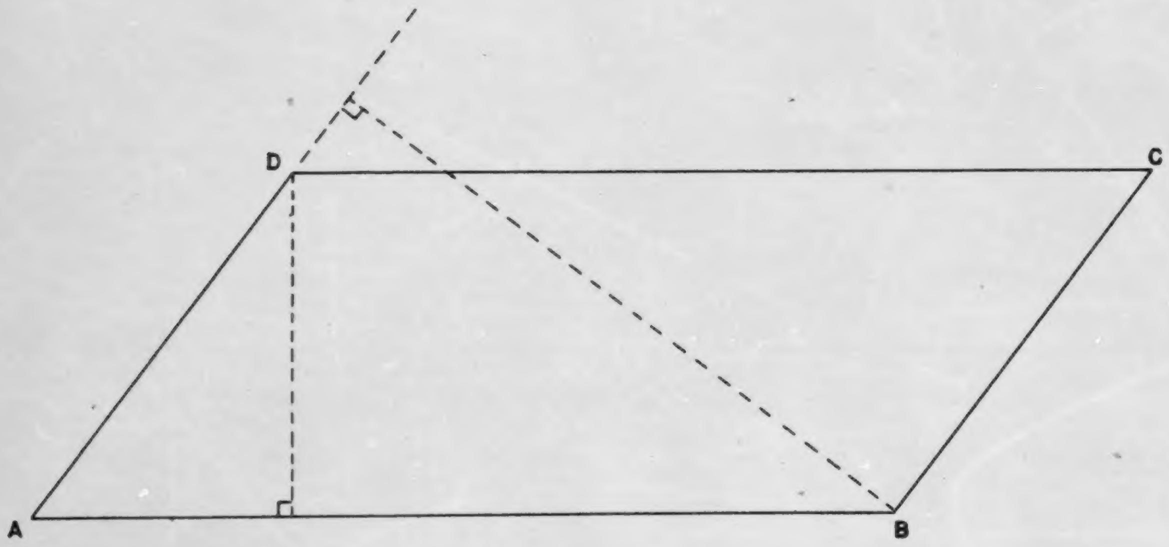


of the two pieces into which the path divided the lot.

5. If b is the number of linear units in the length of the base of a parallelogram and h the number of linear units in the length of the altitude to this base, write a number sentence showing how to find the number A of square units in the area of the interior of the parallelogram.

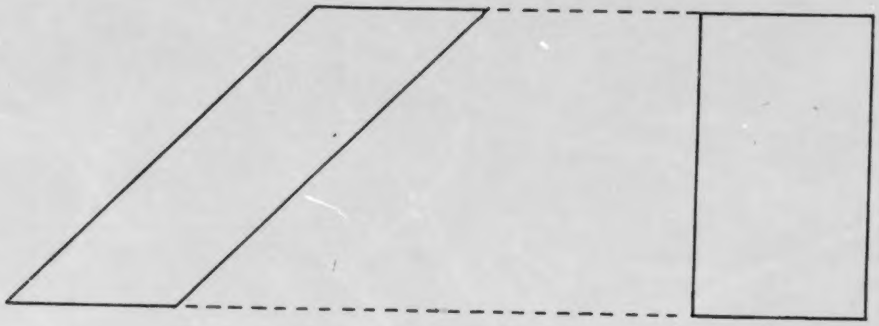
6. If b is the number of linear units in the length of the base of a triangle and h the number of linear units in the length of the altitude to the base, write a number sentence showing how to find the number A of square units in the area of its interior.

7. Measure the bases and altitudes of the parallelogram below and thus find its area in two ways. How well do your results agree? Since measurement is approximate, they may not be exactly the same, but should be close.



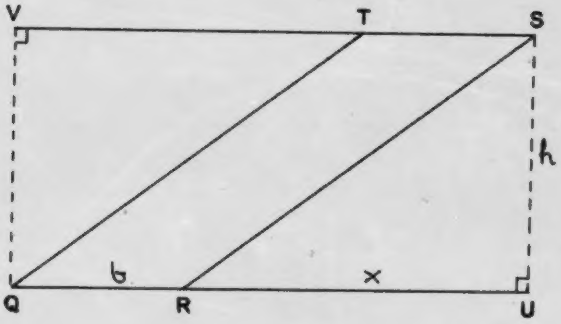
8. Measure the bases and altitudes in the triangle ABC on the next page and thus find the area of this triangle in three different ways. How well do your results agree? As in Problem 7, they may not be identical, but should be fairly close.

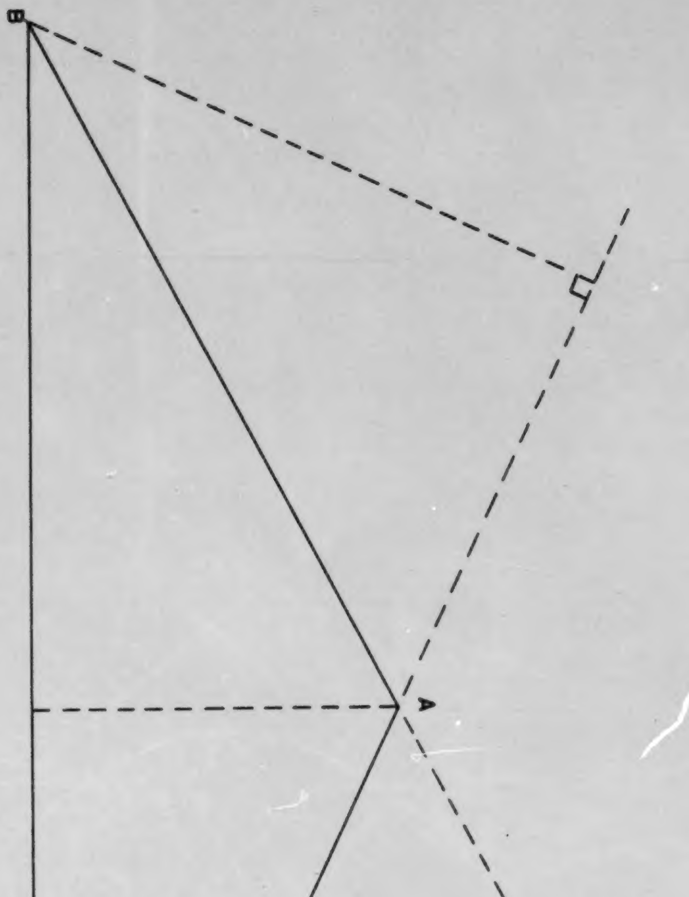
9. BRAINBUSTER. Look at the parallelogram and rectangle below whose bases have equal lengths and whose altitudes also have equal lengths.



Cut the interior of the parallelogram into pieces which can be reassembled to form the interior of the rectangle. (First copy the parallelogram on another sheet of paper.)

10. BRAINBUSTER. Let QRST be any parallelogram not a rectangle, as shown.

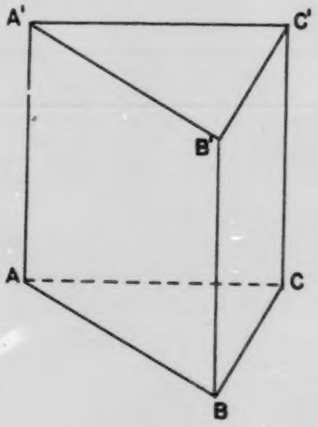



B
A
C

From vertices Q and S where the angles of the parallelogram are acute, draw the perpendiculars $\overline{QV}$ and $\overline{SU}$, giving a rectangle QUSV. Let b be the number of units in $\overline{QR}$, h the number of units in $\overline{US}$, and x the number of units in $\overline{RU}$. Show that the number A of square units of area in the interior of QRST is given by the number sentence $A = bh$. (Hint: The area of the interior of the parallelogram is the area of the interior of the rectangle minus the areas of the interiors of the two triangles.) This discussion justifies the rule for finding the area of a parallelogram for any case.

9-9. Right Prisms

In Chapter 7 you learned about rectangular prisms. Here some other kinds of prisms will be introduced. Let us imagine two triangles of exactly the same size and shape lying in parallel planes. Triangles ABC and A'B'C' in the figure below represent such triangles.

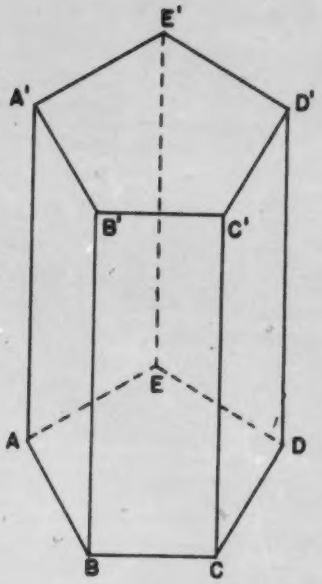


If the segments are drawn joining corresponding vertices, three quadrilaterals (four-sided polygons) are obtained. In this case they are ABB'A', BCC'B', and CAA'C'. If the triangles are so placed with respect to each other that these quadrilaterals are all rectangles, the resulting figure is called a triangular right prism.

The six points A, B, C, A', B', C' are called the vertices of the prism, the various segments shown in the figure are its edges, and the interiors of the two triangular ends and of the three rectangular sides are called its faces. To distinguish them from the interiors of the rectangular sides, the interiors of the two triangular ends are often called the bases of the prism. How many edges, vertices, and faces are there on a triangular prism?

Very likely you have seen solids made from glass whose surfaces are triangular prisms. When held in sunlight such a solid has the effect of bending the light rays to produce the familiar rainbow effect. In fact, such solids are often called prisms.

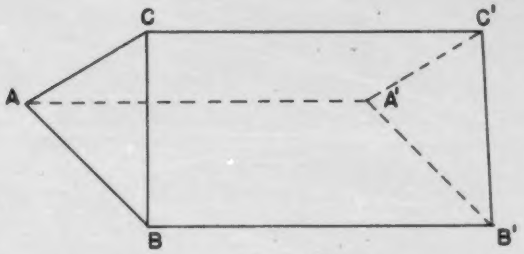
If, in place of using the interiors of triangles for bases, we use the interiors of other polygons, the resulting figures are other kinds of prisms. (Recall the definition of polygon from Section 7.) For example, look at the figure shown on the next page in which the ends are pentagons (five-sided polygons) of the same size and shape in parallel planes, and so related that the quadrilaterals ABB'A', BCC'B', etc., are all rectangles. This figure is called a pentagonal right prism.



In general a right prism is a figure obtained from two polygons of the same size and shape so located in parallel planes that when the segments are drawn joining corresponding vertices of the polygons, the quadrilaterals obtained are all rectangles. The prism consists of the interiors of these rectangles, the interiors of the original polygons, and the segments which bound these interiors. The interiors are called the faces of the prism, the segments are its edges, and the points where two or more edges meet are called vertices. The bases of the prism are the interiors of the original polygons.

As was indicated, the figures described here are called right prisms. Later you will meet more general prisms for which the quadrilaterals mentioned are allowed to be any parallelograms rather than necessarily rectangles. In this chapter, however, we consider only right prisms, and whenever the word prism is used it will mean right prism. The rectangular prisms discussed in Chapter 7 are, of course, simply the right prisms whose bases are the interiors of rectangles. These prisms have a very interesting property. Any such figure can be thought of as a prism in three different ways, since any pair of opposite faces can be used as bases. No other figure can be thought of as a right prism in more than one way.

In the drawings of the prisms on the preceding pages, it has been convenient to show the planes of the bases as horizontal. However, there should be no difficulty in identifying such figures when they occur in different positions. For example, the figure below represents a triangular prism with bases ABC and A'B'C', even though it is shown resting on one of its rectangular faces.



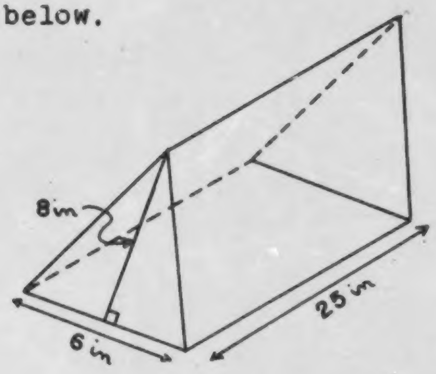
Let us consider now the problem of determining the volume of the interior of a prism. Turn to Section 8 of Chapter 7 and reread the discussion of the volume of the interior of a rectangular prism. This discussion showed that if the area of the base were 12 square units, then by using a total of 12 unit cubes of volume (some of which may be subdivided) we could form a layer one unit thick across the bottom of the prism. If the prism were $3\frac{1}{2}$ units high, it would take $3\frac{1}{2}$ such layers to fill the prism, or a total of $(12)(3\frac{1}{2})$ unit cubes, so the volume is 42 cubic units.

In this discussion it was not necessary to consider the actual shape of the base. In fact the same reasoning applies to finding the volume of the interior of any right prism, no matter what the shape of the base, for the interior of any right prism can be considered as consisting of a series of identical layers piled on each other. We thus obtain the following conclusion:

> The number of cubic units of volume in the interior of a right prism is the product of the number of square units of area in the base and the number of linear units in the height.

In this statement, the term height means the perpendicular distance between the planes of the bases. It is the length of the segments from any vertex of one base to the corresponding vertex of the other base. Notice especially that the height is not measured vertically unless the planes of the bases happen to be horizontal. In the last figure, for example, the height is the length of any one of the segments $\overline{AA'}$, $\overline{BB'}$, or $\overline{CC'}$.

As an example let us find the volume of the interior of the triangular prism shown below.

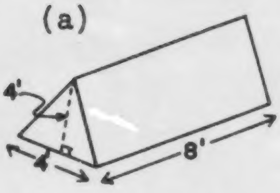


The bases are the interiors of triangles, and by Section 9-8 the number A of square inches in this triangular base is $A = \frac{1}{2}(6)(8) = 24$, so the area is 24 sq. in., but the number of inches in the height of this prism is 25. Thus by the statement above the number of cubic inches in the volume is $24 \cdot 25 = 600$, so the volume is 600 cu. in.

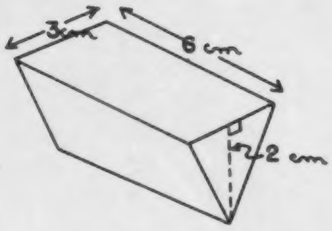
Exercises 9-9

1. Find the number of cubic units of volume for each of the prisms shown below:

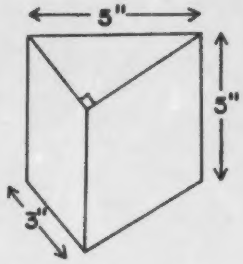
(a)



(b)

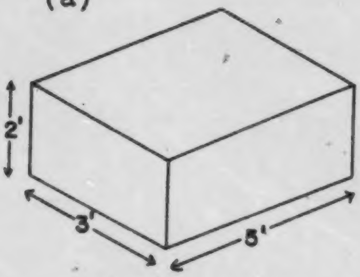


(c)

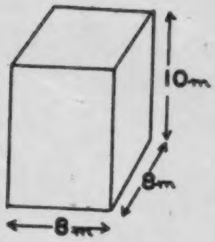


2. Find the number of cubic units of volume for each of the prisms shown below:

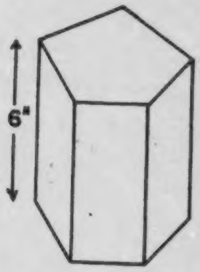
(a)



(b)



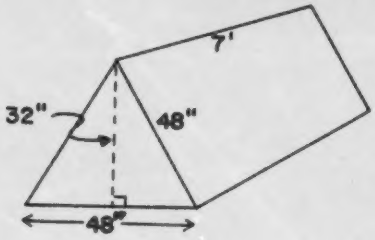
(c)



Area of the interior
of the pentagon is
21 square inches.

3. The columns in front of a building are in the shape of prisms 18 ft. high. The bases are the interiors of hexagons 15 inches on a side. (A hexagon is a polygon with six sides.) If the columns are to be painted, find the number of square feet of surface for each column to be painted. (Notice that the bases--i.e., the ends--are not to be painted.)

4. A pup tent in the shape of a triangular prism is 7 ft. long. The measurements of one end are given in the drawing.



   (a) If this pup tent has both ends and a floor, how many square feet of canvas is used in its construction? (Make no allowance for seams.)

   (b) If the tent has both ends but does not have a floor, how much canvas is used?

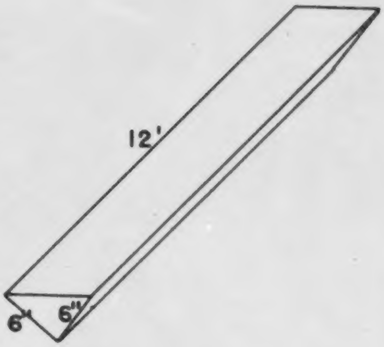
(c) How many cubic feet of air are in the tent?

5. If B stands for the number of square units of area in the base of a prism, and h is the number of linear units in its height, write a number sentence showing how to find the number V of cubic units of volume in the interior of the prism.

6. A container in the shape of a prism is 11 inches high and holds one gallon. How many square inches are there in the base? Do you know the shape of the base? (A gallon contains 231 cu. in.)

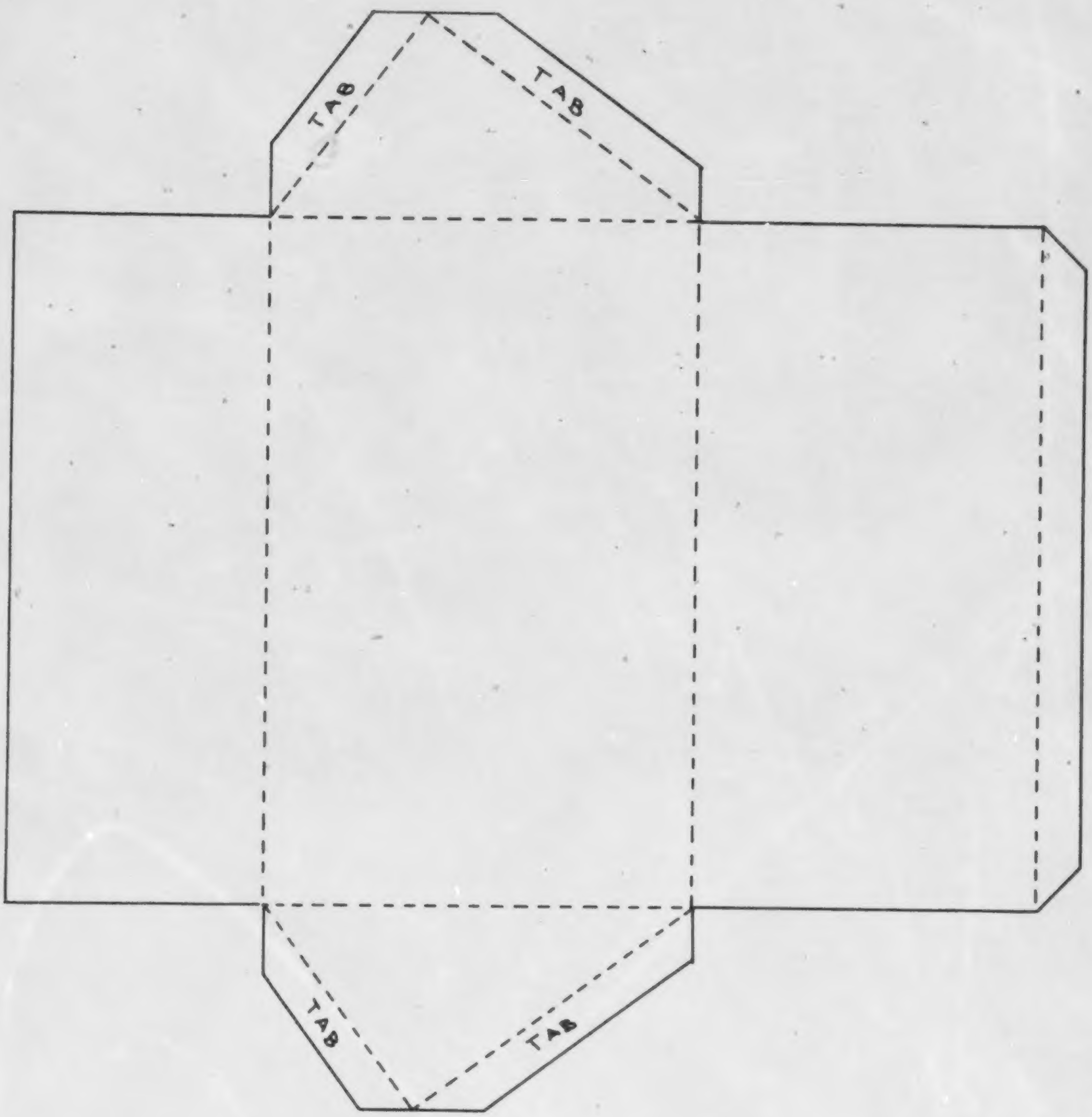
7. A triangular prism has a base which is the interior of a right triangle, with the two perpendicular sides 3 inches and 6 inches. If the prism is 20 inches high, what is the volume in cubic inches?

8. A trough in the shape of a triangular prism is made by fastening two boards together at right angles and putting on ends. If the inside measurements are 6 inches, as shown, and if the trough is 12 ft. long, find how many cubic feet of water it will hold.

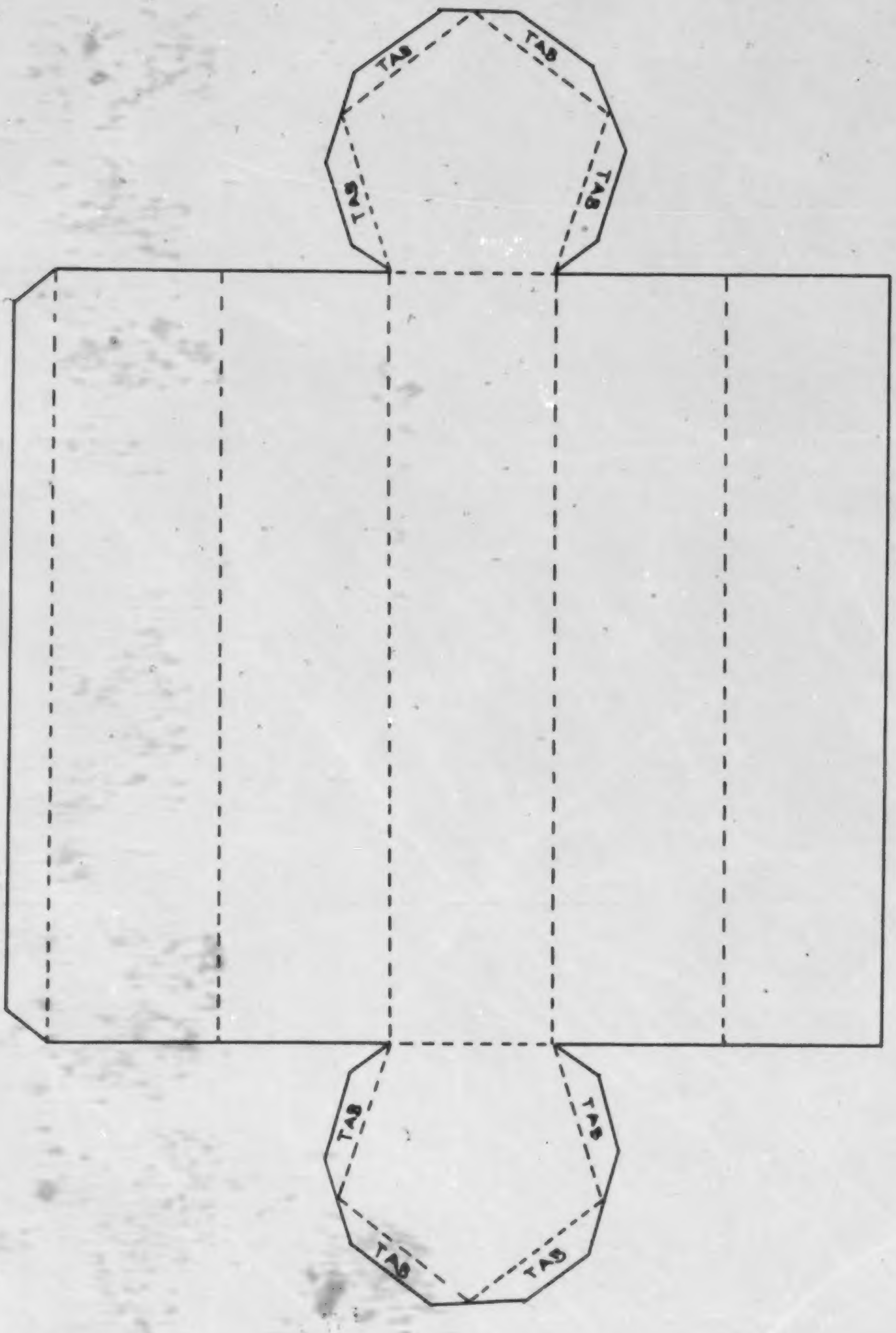


9. Make models of triangular and pentagonal prisms from stiff paper. Either make your own patterns, or use those on the following pages.

10. How many edges, faces, and vertices are there on a triangular prism? a pentagonal prism? a hexagonal prism (6 sides)? an octagonal prism (8 sides)?



Pattern for Pentagonal Prism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9-10. Summary

Chapter 9 dealt largely with some of the relationships which exist between lines on a plane. Section 1 dealt with properties of two lines in a plane. Here you studied pairs of angles called vertical angles and observed the following property:

Property 1. When two lines intersect, the two angles in each pair of vertical angles formed have equal measures.

In Section 1 adjacent angles and supplementary angles were also considered.

Section 2 dealt with properties of three lines in a plane, introducing the ideas of transversals and pairs of corresponding angles. Do you recall the different kinds of figures that may be formed when three lines are drawn on a plane?

In Section 3, information from Sections 1 and 2 was used to investigate two important properties.

Property 2. When a transversal intersects two lines in the same plane, and a pair of corresponding angles have different measures, then the lines intersect.

Property 2a. When a transversal intersects two lines in the same plane and a pair of corresponding angles have equal measures, then the lines are parallel.

In the language of sets, Property 2 refers to two lines whose intersection is not the empty set, while Property 2a refers to two lines whose intersection is the empty set.

Converse statements were considered in Section 4. Examples were given to show that the converse of a true statement might be true, and that the converse of another true statement might be false. You were asked to write a converse for Property 2 and for Property 2a, as follows:

Converse of Property 2. If two lines in a plane intersect, then if they are intersected by a transversal any pair of corresponding angles have different measures.

Converse of Property 2a. If two lines in a plane are parallel, then if they are intersected by a transversal any pair of corresponding angles have equal measures.

You will remember that you found these converses were both true.

Section 5 introduced the names for different sets of triangles; isosceles, equilateral, and scalene. Scalene triangles are those having no two sides with the same measure. Isosceles triangles are those having at least two sides equal in length, while equilateral triangles are the special set of isosceles triangles for which all three sides have the same length. In this section we discovered the following property of isosceles triangles, and found that its converse is also true.

Property 3. If two sides of a triangle have the same length, then the angles opposite these sides have equal measures.

Converse of Property 3. If two angles of a triangle have equal measures, then the sides opposite these angles have the same lengths.

In Section 6 we observed:

Property 4. The sum of the measures, in degrees, of the angles of any triangle is 180.

You obtained this property by using the inductive method of reasoning when you tore the corners from a triangular region and placed them as adjacent angles. You also learned to prove this property by the deductive method when you used other properties to show that

$$m(\angle x) + m(\angle y) + m(\angle z) = 180$$

where x, y, and z stand for the angles of a triangle.

The parallelogram studied in Section 7 belongs to the special set of polygons called quadrilaterals. Do you remember what quadrilateral means? A parallelogram belongs to the special set of quadrilaterals whose opposite segments lie on parallel lines. This was included in Property 5.

Property 5. A pair of opposite sides of a parallelogram are parallel and equal in length.

You also learned in this section that the shortest segment from a point to a line is perpendicular to the line, and that two parallel lines have a constant distance apart.

You used certain properties of parallelograms in Section 8 in finding the areas of a parallelogram and a triangle:

(a) The number of square units of area in the interior of a parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to this base.

(b) The number of square units of area in the interior of a triangle is half the product of the number of linear units in the base and the number of linear units in the altitude to this base.

Section 9 dealt with volumes of interiors of right prisms, and was an extension of the work with rectangular prisms in Chapter 7. Here you learned to find the volume of the interior of any right prism:

> The number of cubic units of volume in the interior of a right prism is the product of the number of square units of area in the base and the number of linear units in the height.

9-11. Historical Note

Some of the geometric ideas we study about in Chapter 9 were discovered by the Egyptians and Babylonians almost 4,000 years ago. For example, they knew how to find the area of a triangle and used this knowledge in surveying and measuring fields.

Thales, mentioned in Section 2, is credited with the discovery that the measures of the base angles of an isosceles triangle are equal. There is some evidence that Thales also knew that the sum of the measures in degrees of the angles in a triangle is 180.

There were many other famous Greek mathematicians. Their work made ancient Greece famous as the "Cradle of Knowledge." We will discuss only a few of these men. Pythagoras (569 ? B.C. - 500 B.C.) organized schools at Croton in southern Italy which contributed to further progress in the study of geometry. You will learn about some of the discoveries credited to him next year. Euclid (365 ? B.C. - 300 ? B.C.) became famous by writing one of the first geometry textbooks called the Elements. This textbook has been translated into many languages. It has been used in teaching geometry classes for some 2,000 years without much change. Its form has been somewhat modernized to fit present needs. All of the properties we have studied in this chapter may be found in the Elements.

From the 7th century until the 13th century very little progress was made in mathematics. From the 13th century, however, the study of geometry and other mathematics spread rapidly throughout Europe. Mathematicians began to examine new ways of studying elementary mathematics. You will learn about the work of men such as Rene Descartes (1596 - 1650, France); Blaise Pascal (1623 - 1662, France); Pierre Fermat (1601 - 1665, France); Karl Friederich Gauss (1776 - 1855, Germany); and others as you continue your studies of mathematics.

At the present time many new mathematical discoveries are being made in many parts of the world. There are still many important unsolved problems in geometry. As one example, suppose you have a lot of marbles all exactly the same size. How should you pack them in a large container in the best possible way, that is, so as to get as many marbles as you can into the given volume? Nobody knows for sure. We know a very good way to pack the marbles, but no one has ever proved that it is the best possible way.

Originally geometry meant a study of earth measure. The word "geometry" comes from two Greek words, "geo" meaning earth and "metry" meaning measure. Through the years geometry has come to mean a study of such elements as points, lines, planes, space, surfaces, and solids. These elements are used to describe the shape, size, position, and relations among objects in space.

CHAPTER 10

CIRCLES

One of the most common and most useful simple closed curves is the circle. No matter where you are, it is probable that you will be able to see one or more objects which suggest a circle. Find some in your classroom now, and find as many as you can in your home. Perhaps there may be a traffic circle near your school. Do you think a circle is a good figure to use in this way?

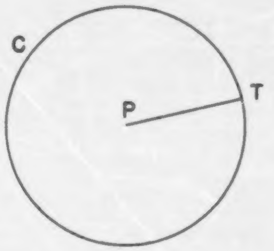
In Chapter 4 you learned about simple closed curves, and in later chapters you studied some of the characteristics of several kinds of simple closed curves, such as parallelograms, rectangles, and triangles. In this chapter you will study some of the properties of a circle as a mathematical figure. You know that a line or a simple closed curve may be thought of as a set of points. First let us see how we may describe a circle as a set of points.

10-1. Circles and the Compass

Choose a point on a sheet of paper and label it point P. Then use your ruler to mark at least ten points, each at a distance of 3 inches from P. What figure do these points suggest? If you have located the ten points in widely different directions from P, they should suggest a circle.

To draw the complete circle, use a compass. Most of you are already familiar with the compass as a device for drawing circles. (An instrument for indicating which direction is north is also called a compass, but we are not concerned with that kind of compass here.) To draw the circle with a compass, adjust the arms of the compass so that the distance between the sharp point and the pencil tip is 3 inches. Put the sharp point of the compass at the point P. Then pivot the compass about this point, making the pencil tip trace a curve. When it returns to the starting point you have a circle. It is clear that a circle is a simple closed curve. You drew it in the manner described in Chapter 4 for drawing a simple closed curve.

In your circle, point P is the center of the circle. Choose one of the points you located at a distance of 3 inches from P, and label it point T. Draw the segment $\overline{PT}$. Your figure should look like the one below, except that yours should be larger.) This circle may be called "circle P", meaning a circle which has as center point P. Sometimes we name the curve by a letter, as "circle C."



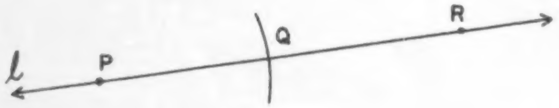
The segment $\overline{PT}$ is called a radius of the circle. A radius is any line segment which joins the center P to a point on the circle. Draw a second radius and call it $\overline{PY}$. When we speak of more than one radius we say "radii". $\overline{PT}$ and $\overline{PY}$ are radii of the circle. How many radii may there be?

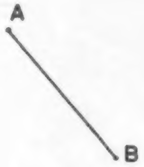
We use the word "radius" in another way too. It means the distance from the center to any point on the circle. The radius of the circle that you drew is 3 inches. There is just one distance which is "the" radius of a circle, but "a" radius may be any line segment with the center and a point on the circle as its endpoints.

We can now describe a circle as a set of points. The circle with center P and radius r units is the set of all points in a plane at distance r units from P.

Look at your drawing again. Choose any new point in the plane and call it point Q. Draw the ray PQ. On the ray PQ measure a distance of 3 inches from P. The point you obtain will be on the circle. Call the point S. Is $\overline{PS}$ a radius of the circle?

The compass has a second use. It is used for transferring distances. Draw a line ℓ and label two points P and R on .
Draw a segment $\overline{AB}$ anywhere on your paper. Without using a ruler, we wish to find a point Q on PR such that the length of $\overline{PQ}$ is the length of $\overline{AB}$.





Take your compass and adjust its arms so that when you put the sharp point at A, the pencil tip is at B. Then, without changing the opening of the arms, lift the compass and place the sharp point at P. Draw a small part of a circle crossing the ray $\overrightarrow{PR}$, still without changing the opening of the arms of the compass. Call the point of intersection Q. Then the length of $\overline{PQ}$ is the length of $\overline{AB}$.

Exercises 10-1

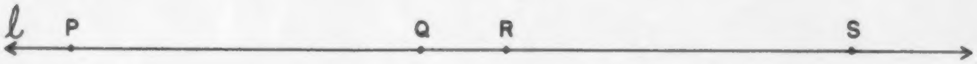
In the problems below you will have practice in using your compass to draw circles and to transfer distances. Read the directions carefully, and label each point, circle or line segment before you go on to the next direction.

1. (a) Label a point P on your paper. Draw a circle with center P and radius 7 cm. Call it circle C.

   (b) Label Q a point of circle C. Draw a circle with center Q and radius 3.5 cm. Draw another circle with center P and radius 3.5 cm.

   (c) What does the intersection of the last two circles seem to be?

2. In this problem, you are to transfer some distances to your paper from the figure below.



   (a) Draw a vertical line m on your paper, and label a point A on the lower part of the line.

   (b) Use your compass to locate point B above A on line m , so that AB = PQ. (Note that "AB", with no symbol above it, means the measure of segment $\overline{AB}$.)

   (c) Locate point E above A on line m , so that AE = PR.

   (d) Locate point F above A on m so that AF = PS.

(e) With B as center and $\overline{BF}$ as radius, draw a circle.

(f) If your drawing is accurate, there will be two labeled points on the circle. Name the point.

3. (a) Draw two intersecting lines which are not perpendicular, and call the lines ℓ_1 and ℓ_2. (Notice that we have named both lines with the same letter, but have written numerals after the letters, and a little lower. You will recall that such numerals are called "subscripts". ℓ_1 is read "ℓ sub-one", or sometimes just "ℓ one.")

(b) Call the point of intersection of ℓ_1 and ℓ_2 point B. With B as center and radius 1 inch draw a circle.

(c) Label the intersection of the circle with ℓ_1 points R and S, and the intersection with ℓ_2 points T and Y.

(d) Draw $\overline{RT}$, $\overline{RY}$, $\overline{ST}$, and $\overline{SY}$. What kind of figure does RTSY seem to be?

4. Draw a line ℓ and label points X and Y about 1 inch apart on ℓ.

(a) Draw the circle C_1 which has its center at X and passes through Y.

(b) Draw the circle C_2 which has its center at Y and passes through X.

(c) Label Z as the other intersection of circle C_2 and line ℓ.

(d) Draw the circle C_3 which has its center at Z and passes through X.

(e) What is $C_1 \cap C_3$?

(f) What is $C_2 \cap C_3$?

*5. (a) Draw two intersecting lines ℓ_1 and ℓ_2. Label as B the intersection of ℓ_1 and ℓ_2.

(b) Label A s point of ℓ_1 and C a point of ℓ_2, so that the length of $\overline{BA}$ is not the length of $\overline{BC}$.

(c) Use your compass to mark a point A_1 on $\overleftrightarrow{AB}$ such that A_1 is not on $\overrightarrow{BA}$ and $A_1B = AB$.

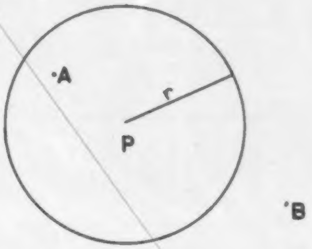
(d) Mark a point C_1 on $\overrightarrow{CB}$, but not on BC, such that $BC_1 = BC$.

(e) Draw $\overline{AC}$, $\overline{AC_1}$, $\overline{A_1C}$, and $\overline{A_1C_1}$. What kind of figure does ACA_1C_1 look like?

(f) How many triangles are represented in your total figure?

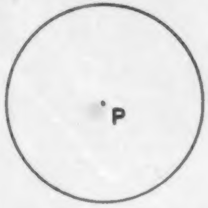
10-2. Interiors and Intersections

A circle is a simple closed curve. Consequently it has an interior and an exterior. Suppose we have a circle with center at P and with radius r units. A point such as A inside the circle is less than r units from the center P, while a point outside the circle, like B, is more than r units from P. Thus is is easy, in the case of the circle, to describe precisely what its interior and its exterior are. The interior is the set of all points at distance less than r units from P. The exterior is the set of all points at distance greater than r units from P.

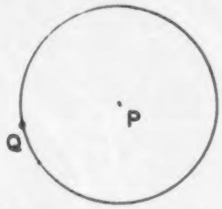


Given two sets, we have frequently worked with the notion of their intersection. A circle is an example of a set of points. Consequently we may raise questions about intersections involving circles.

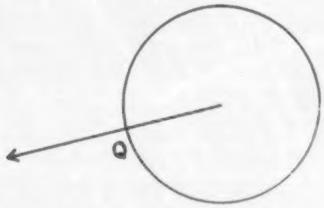
Let us choose a point P and any length. Let us draw the circle with center at P and with the selected length as its radius. (With your compass, make the drawing on a piece of paper. Your picture should look like the figure.)



Next let us choose any point Q on the circle. (After we make our choice, the diagram may appear as shown.)



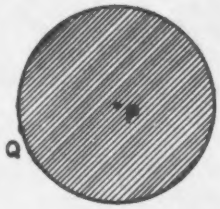
We now ask two questions. How many points on the circle are also on the ray $\overrightarrow{PQ}$? What is the name of each? To answer these questions, you may find it much easier to draw the ray on your paper and then inspect the picture.



(In general you should acquire the habit of putting on paper whatever you need in order to understand the ideas you are studying. Notice how we have done this, step by step, in our discussion. Follow this suggestion throughout the remainder of the chapter.)

With the same situation as in the previous paragraph, how many points on the circle are also on the ray $\overrightarrow{QP}$? (Did you draw the ray $\overrightarrow{QP}$ before you attempted to answer?) Do you feel the need to label one or more points which we have not yet named? Can you describe carefully (in words) the location of each extra point which you believe ought to be named?

Now shade lightly the interior of the circle. What is the union of the circle and its interior?



(Did you need to look up the word "union" in Chapter 4 in order to answer this question?) No point of the circle also lies in the interior of the circle; sometimes we say that the intersection of the circle and its interior is the empty set. The set of all points which are either on or inside the curve is the union of the circle and its interior. Another way of thinking about the union of the circle and its interior is the following: the union is the set of all points whose distance from the center P is either the same as or less than the __________ of the circle (what word belongs in the blank?) Let us use the letter D to represent the union of our circle and its interior.

What is the intersection of the set D and the line $\overleftrightarrow{PQ}$? In what way is the set $D \cap \overleftrightarrow{PQ}$ quite different from the intersection of $\overleftrightarrow{PQ}$ and the circle?

In Figure 10-2, the point P is the center of the circle, the four points A, F, B, G lie on the circle, the intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$ consists of the point P, and the two lines are perpendicular. Copy the diagram on a piece of paper. (The steps in copying are as follows: with compass draw the circle, mark the center P, draw the line AB through P, use a protractor to obtain a right angle, draw FG perpendicular to $\overleftrightarrow{AB}$, label the points.) Shade the halfplane which contains F and whose boundary is the line $\overleftrightarrow{AB}$. (A colored pencil or crayon is good for shading; but if you do not have one handy, just shade lightly with your ordinary pencil.) Let us call this halfplane H.

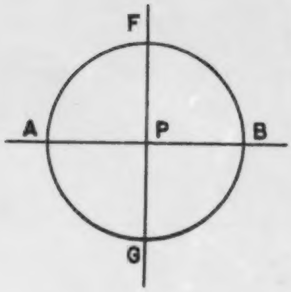


Figure 10-2

In the same picture on your paper, indicate very clearly the intersection of the halfplane H and the circle. Does A belong to this intersection? Does G? Does F? Does P? Can you count how many points belong to this intersection? Choose (and label as M and N on your diagram) two points of this intersection which also belong to the interior of the angle BPF. Similarly choose one point (call it K) which lies on the circle and also in the interior of the angle APF. Identify the intersection of the circle and the angle MKN. Then identify the intersection of the interior of the circle and the interior of the angle MKN.

Exercises 10-2

1. Let C be a circle with center at P and radius r units. Let S be any other point in the same plane. (Are you drawing the figure, step by step, as the problem is describing it?)
   (a) How many points belong to the intersection of the circle C and the ray $\overrightarrow{PS}$?

(b) How many points belong to the set $C \cap \overrightarrow{PS}$?

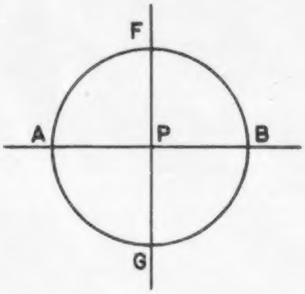
(c) Do your answers to parts (a) and (b) depend upon where you chose the point S?

(d) How many points belong to the set $C \cap \overline{PS}$? Does this answer depend on the choice of S? If so, how?

2. In a plane, can there be two circles whose intersection consists of just one point?

3. Choose two points and label them P and Q. Draw two circles with center at P such that Q is in the exterior of one circle and in the interior of the other. Label the first circle C and the second circle D.

4. The diagram at the right is a copy of Figure 10-2. We let the halfplane on the F-side of $\overleftrightarrow{AB}$ be called H. Let J be the halfplane on the B-side of $\overleftrightarrow{FG}$.



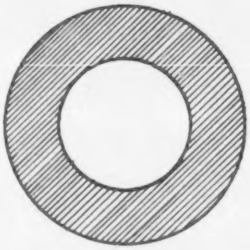
(a) What is the set $H \cap J$?

(b) What is the intersection of all three of the sets, J and H and the circle?

(c) Does the diagram suggest to you that the circle has been separated into what might be called quarters? If so, can you describe several of these portions?

(d) Can you find a portion which might be called half of the circle? Can you describe it in the language of intersections or unions of sets? Can you identify several such portions? More than two?

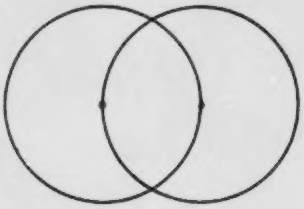
5. Choose two distinct points P and Q. Draw the circle with center at P and with the segment $\overline{PQ}$ as a radius. Then draw the circle with center at Q and with P on the circle.
   (a) What is the intersection of these two circles?
   (b) Can you draw a line which passes through every point of the intersection of the two circles? More than one such line? Why?
   (c) In your picture shade the intersection of the interiors of the two circles. (If you have a colored pencil handy, use it for shading; if you do not, use your ordinary pencil and shade lightly.)
   (d) (In this part, use a different type of shading or, if you have one handy, use a pencil of a different color.) Shade the intersection of the interior of the circle with center P and the exterior of the circle with center Q. (Before doing the shading of the intersection, did you find it helpful to mark separately the two sets whose intersection is desired?)
   (e) Make another copy of the picture showing the two circles, and on it shade the union of the interiors of the two circles.
6. The two circles are concentric, that is, they lie in the same plane and have the same center.
   (a) Describe the intersection of the two circles.



(b) Give a word description of the shaded region. (You should use in your response such words as "intersection," "interior," "exterior.")

*7. Refer to the figure in Problem 6. The intersection of the exteriors of the two circles may be more simply described. How?

*8. The center of each circle lies on the other circle. Copy the figure on your paper. Shade the union of the exteriors of the two circles.



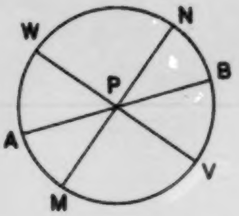
*9. Choose two distinct points P and Q. Draw the line $\overleftrightarrow{PQ}$. Draw the circle with center at P and with Q on the circle. Draw the circle with center at Q and with $\overline{QP}$ as a radius. Draw a line which passes through each point of intersection of the two circles, and call the line ℓ. By inspecting the diagram, make an observation about a relationship between the line ℓ and the line $\overleftrightarrow{PQ}$. Use your protractor to check your observation.

10-3. Diameters and Tangents

Closely associated with the word "radius" is the word "diameter." We have seen that the word "radius" can be used in two different ways. By way of review, a radius of a circle is one of the segments joining a point of the circle and the center; the length of one of these segments is the radius of the circle.

A diameter of a circle is a line segment which contains the center of the circle and whose endpoints lie on the circle. Just as a circle has many radii, so it has many diameters. For the circle represented in the figure, three diameters are shown, namely

$\overline{AB}$, $\overline{MN}$, and $\overline{VW}$. How many radii are shown in the figure?



A set of points which is a diameter may be described in another way. A diameter of a circle is the union of two different radii which are segments of the same line. How does the length of a diameter compare with the length of a radius?

The length of any diameter of a circle we call the diameter of the circle. The diameter is a distance, and the radius is a distance. The measure of the diameter is how many times the measure of the radius?

If we choose any unit of length, and if we let r and d be the measures of the radius and the diameter (of the same circle), then we have the important relationship: $d = 2 \cdot r$. What replacement for the question mark makes the sentence $r = ?d$ a true statement?

The line and the circle in the figure remind us of a wheel (without flange) resting on a track. How many points on the circle are also on the line? Name each such point. We may say that the line is tangent to the circle, and the single point T of their intersection may be called the point of tangency.

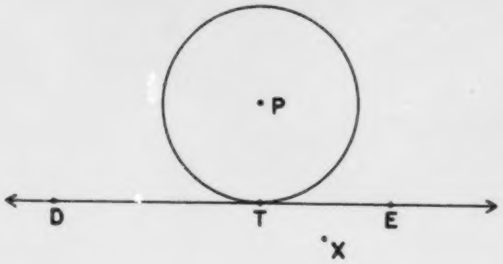
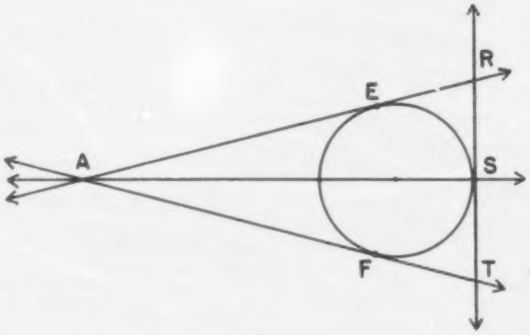


Figure 10-3

Class Questions

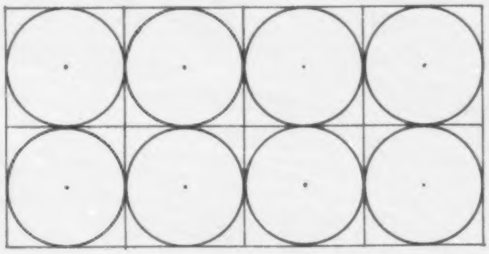
1. In Figure 10-3, the line DE separates the plane DPX into two halfplanes. How can you describe the intersection of the halfplane containing X and the circle? What is the intersection of the halfplane containing P and the interior of the circle?
2. Can you draw a circle and a line in the same plane such that their intersection is the empty set?
3. Can you draw a circle and a line such that their intersection contains four points?
4. How many lines are shown in the figure? How many of these lines are not tangent to the circle? Name each point of tangency shown.



5. Suppose that we are given a circle and a point on the circle. How many radii of the circle contain the given point? How many diameters of the circle contain the given point?

Exercises 10-3

1. How many illustrations of the tangent notion do you find in this pattern?

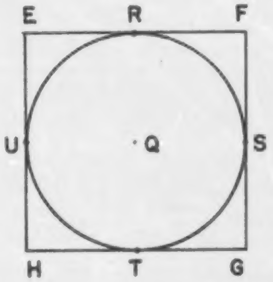


2. During the next week, everywhere you go, keep watch for all the examples you can find which represent the idea of a circle and a line tangent to the circle, that is, of a line and a circle whose intersection consists of a single point.
3. The diameter of a certain circle is 42 cm. How many centimeters from the center of the circle is a point on the circle?
4. Draw a circle C with center at the point P. Draw three diameters of C. Draw a circle with center at P whose radius is equal to the diameter of C.
5. (Warning: this problem requires very careful handling of the compass.)
   (a) Near the middle of a sheet of paper, mark a point Q.
   (b) Draw a large circle C with center at Q, and keep the opening of the compass arms fixed after the drawing is completed.
   (c) Mark a point U on the circle C.
   (d) With the compass, find a point V on the circle C such that $\overline{UV}$ has the same length as $\overline{QU}$.
   (e) Again with the compass, find a third point W on C such that $\overline{VW}$ has the same length as $\overline{QU}$.
   (f) Continue on around, locating points X, Y, Z on C (use compass three times) such that the length of each of the segments $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$ is the same as the radius of the circle.
   (g) Now compare the length of the segment $\overline{ZU}$ with the radius of the circle.

   If you had a fine quality compass and if you were able to perform the construction with great care, the simple closed curve UNWXYZ would represent a "regular hexagon."

(h) What name might you give to each of the segments $\overline{UX}$, $\overline{VY}$, and $\overline{WZ}$?

6. In the diagram the point Q is the center of the circle and the center of the square EFGH.



(a) What is the intersection of the circle and the square?

(b) What is the intersection of the line $\overleftrightarrow{GH}$ and the circle?

(c) What new name have we given to the point T?

(d) How many other lines (segments of which are represented in the figure) are tangent to the circle?

(e) Name all the points of tangency you can.

7. On your paper make a sketch of the diagram for Problem 6. (A careful drawing is not needed here.) On your copy, draw the quadrilateral RSTU.

(a) How many sides of this quadrilateral are segments of lines tangent to the circle?

(b) What is the intersection of the interior of the circle and the exterior of the square EFGH?

(c) Describe carefully the intersection of the exterior of the circle and the interior of the square EFGH.

(d) What is the intersection of the interior of the circle and the exterior of the quadrilateral RSTU?

*8. Refer again to the carefully drawn figure in Problem 6.

(a) Do you think that all three of the points R, Q, T lie on one line?

(b) Do you believe that the three points U, Q, S lie on one line?

(c) Make an estimate of the size of the angle QTG. Can you check your estimate with a protractor?

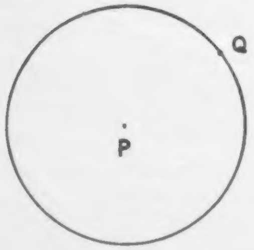
(d) Make an observation about a relationship between the line QS and the line FG. Can you check your observation with a protractor?

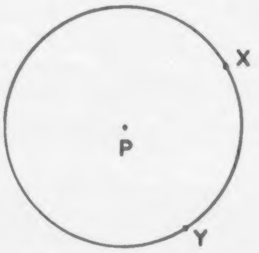
*9. Draw any circle and any line tangent to the circle. Draw also the line which joins the center of the circle and the point of tangency. Do you believe there is an important relationship between these two lines? If so, what is it?

*10. We know from the description of a circle that all radii have the same length. How do we know that all diameters of a given circle have the same length?

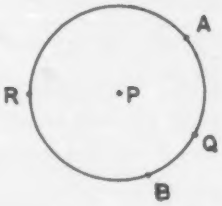
10-4. Arcs and Central Angles

In Chapter 4 we learned that a single point on a line separates the line into two pieces. This idea of separation led us to see that on a line a single point determines two half-lines.

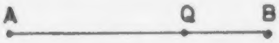


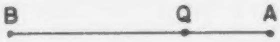


Does a single point on a circle separate the whole circle into two portions? A single point such as Q in the figure above does not separate the circle into two parts. What does it take to separate a circle? In the figure to the right above do points X and Y separate the circle into two parts? Notice in the figure to the right how the two points A and B separate the circle into two parts, one of which contains R and the other of which contains Q. No path from R to Q along the circle can avoid at least one of the points, A or B. In summary we would say that it takes two different points to separate a circle into two distinct parts.

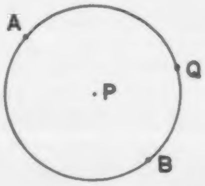


Recall that if three different points A, Q, and B are points on a line as in the figures below, Q would be considered "between" A and B. This is tied in with the separation properties of a point on a line.





This notion may seem so obvious that you wonder why we even bothered with it. Let us consider a different situation.

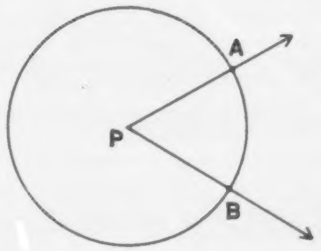


In the figure above a confusing variety of observations could be made. On the circle (or on any simple closed curve for that matter) Q could be considered between A and B. However if we think of the clockwise path from Q around to A, we could say that B is between Q and A. Can you look at the diagram and interpret A as being between B and Q?

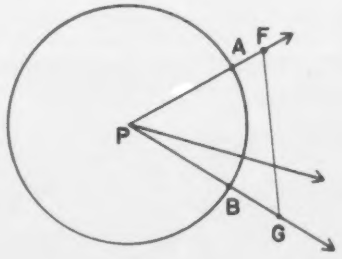
On a line "betweenness" and "separation" are closely related. Contrasted with this is the fact that on a circle, or other simple closed curve, a single point does not separate the curve into two parts. This is related to an unsatisfactory notion of betweenness on the whole circle.

In our earlier discussion of geometry we have found it useful both to talk about and to use segments of lines. It will often be just as necessary for us to consider portions of a circle, or arcs.

In the figure to the right the points A and B separate the circle with center at P into two portions. Each of these pieces together with A and B is an arc. A and B are called end-points of the arc. We must keep in mind that any two distinct points on a circle determine two different arcs having these two points as end-points.

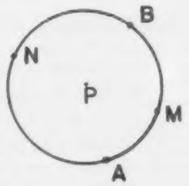


Do arcs have any properties similar to the properties of line segments? In the figure to the side consider the shorter of the two arcs determined by A and B. It is the arc which is associated with the interior of ∠ APB. $\overline{FG}$



connects a point F of PA with a point G of PB. Observe that there is a natural one-to-one correspondence between the set of points of the arc and the set of points of the segment. Rays with end-points at P which intersect $\overline{FG}$ can be used to establish this one-to-one correspondence. A point of segment $\overline{FG}$ and a point of the arc correspond to each other if they are both on a ray with end-point at P.

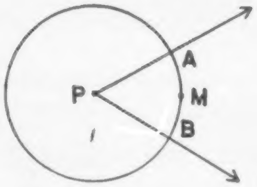
Here we have established a natural one-to-one correspondence between the set of points of an arc and the set of points of a segment. This relationship should give us a firm basis for relating some of the properties of portions of circles with those of line segments. It can be seen that just as a point may separate a segment into two pieces, anarc may also be separated into two portions by a point. Thus the fact that one point may be "between" two other points on an arc is not true with a whole circle. As with a segment, an arc has a "starting" point and a "stopping" point, the two end-points.



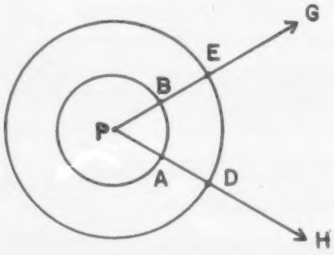
With only two points on a circle, A and B, we cannot easily distinguish the two arcs which these two points determine. Therefore on the figure at the right we have marked and labeled a point between each of the two end-points on each of the arcs. These two points are conveniently located somewhere near the middle of each of the arcs. Now we can use the symbol $\widehat{AMB}$ to represent the arc containing M and $\widehat{ANB}$ to represent the arc which contains N. In place of $\widehat{AMB}$ we can use $\widehat{BMA}$. What other symbol represents the same arc as $\widehat{ANB}$?

Draw a circle and a diameter and label A and B the end-points of the diameter. The two arcs determined by these special end-points are semi circles. Another convenient way of stating the condition on the points is simply that A and B are the end-points of a diameter of the circle. However if the two points A and B and the center do not all lie on the same straight line, then we have an angle associated with the rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$. This angle, $\angle APB$, with its vertex at the center of the circle is called a central angle. Central angles are measured in the same way as other angles and the unit of angle measure is an angle of one degree.

We will often find it necessary to compare one arc with another. Therefore we will find it convenient to devise some method for measuring arcs. You will recall that in measuring angles we used a scale based on a set of 360 rays from a point. These rays form 360 angles whose interiors are all the same size. (Most protractors show just half this scale.) If this set of rays is drawn from a point which is the center of a circle the sides of each one-degree angle determine an arc. The sides of the 360 angles determine 360 such arcs each of which is called "one arc-degree." In the figure at the side the measure of arc $\widehat{AMB}$ in arc degrees is equal to the degree measure of the corresponding central angle. For instance if $m(\angle APB) = 80$ then $m(\widehat{AB}) = 80$, where we are thinking of a special kind of unit in the latter case, the arc degree. Its symbol ($^\circ$) is exactly the same as that for the angle degree.

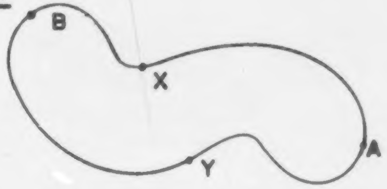


On the set of numbered rays of the angle scale the numbers 0 and 180 correspond to opposite rays, that is, rays which lie on the same line and have the same end-point. The union of these two rays forms a line whose intersection with the circle is the pair of end-points of a diameter. These two points determine a special arc, a semi circle, which we have mentioned earlier in the chapter. Thinking of this angle scale we can think of the semi circle having an arc degree measurement of 180° since it would consist of 180 one degree arcs. Corresponding to the semi circle (a special kind of arc) is a special kind of central angle with a measurement of 180°. Some people find it convenient to speak of the central angle of 180° as a "straight angle". (Why does "straight angle" not agree with our definition of angle?)



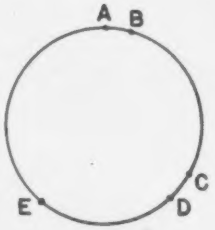
In the figure above are shown two concentric circles (concentric circles were defined in Problem 6 of Exercises 10-2.) Arcs $\widehat{AB}$ and $\widehat{DE}$ must have the same arc degree measure because they share the same central angle, $\angle GPH$. You will recall that the degree measure of an arc is identical with the degree measure of its corresponding central angle. However $\widehat{DE}$ appears longer than $\widehat{AB}$. Focus your attention and thought on the fact that in this case we are talking about the distances along two arcs as being different. Two arcs may have the same arc degree measure but have different lengths. The reason will be apparent when you study Section 5 of this chapter.

Exercises 10-4



1. In the figure at the right the pair of points A and B separate the points X and Y. Which points, if any, do the pair of points listed below separate?

   (a) B, Y (b) A, Y (c) X, Y (d) A, X



2. In the figure at the right determine with the use of your protractor the measure of the following arcs. Indicate your results with correct use of symbols, for example $m(\widehat{AB}) = 15$.

   (a) $\widehat{ABC}$ (b) $\widehat{ABCD}$ (c) $\widehat{DE}$ (d) $\widehat{BCD}$ (e) $\widehat{CDE}$

3. Construct a circle with radius approximately $1\frac{1}{2}$ inches. In this exercise mark off the points in a counter-clockwise path around the circle after starting anywhere on the circle with A. Mark off and label arcs with the following measures:

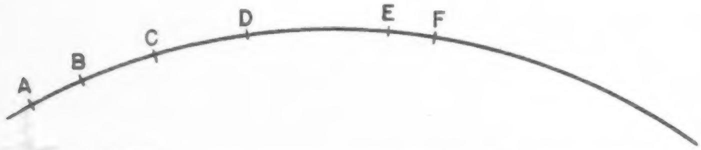
   (a) $m(\widehat{AB}) = 10$ (b) $m(\widehat{AC}) = 45$ (c) $m(\widehat{BD}) = 50$

   (d) $m(\widehat{DF}) = 170$ (e) What is $m(\widehat{BC})$?

4. Demonstrate a one-to-one correspondence between the sets of points on the two semi circles of a given circle which are determined by a diameter.

5. How many arc degrees are in a quarter of a circle? one-eighth of a circle? one-sixth of a circle? three-fourths of a circle?

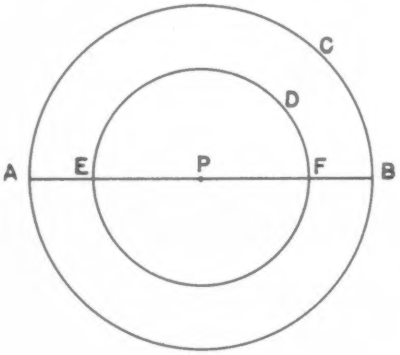
6. Draw a circle with a radius of 2 inches. Starting at any point on the circle use a compass with the same setting (2 inches) to describe a series of equally spaced marks around the circle. CAUTION. Be careful that your compass setting does not change.
   (a) Do you get back to exactly the same point where you started?
   (b) How many arcs are marked off?
   (c) What is the measure of the central angle of any of these arcs?
   (d) What is the degree measure of any one of these arcs?



7. Given the arc $\overset{\frown}{ABCDEF}$, or more briefly AF determine the following:
   (a) $\overset{\frown}{AC} \cap \overset{\frown}{BD}$ (b) $\overset{\frown}{AF} \cap \overset{\frown}{DF}$ (c) $\overset{\frown}{AD} \cap \overset{\frown}{CF}$
   (d) $\overset{\frown}{CD} \cap \overset{\frown}{DE}$ (e) $\overset{\frown}{DF} \cap \overset{\frown}{AE}$

10-5. Length of a Circle

In the figure, circle C and circle D have the same center, P. Name a diameter of circle C; of circle D. Which circle has the longer diameter? Which circle seems to be the longer?



Apparently the longer circle has the longer diameter. This suggests that there may be some definite relation between the length of a circle and the length of its diameter. (The length of a circle is called its circumference.)

We shall use two methods to study the relation of the length of a circle to the length of its diameter. Problem 1 in Exercises 10-5a below gives you directions for Method I, and Problem 2 gives you directions for Method II.

Exercises 10-5a

1. (Method I.)

   (a) Choose three circular objects in your home, and use a tape measure to find the length of each circle. (If you do not have a tape measure, use a piece of firm string, and then measure the string.)

   (b) Measure the diameters of the same three circular objects. Since it may be hard to locate the center of the circle, measure the diameter several times to obtain as good a measure as possible.

   (c) Arrange your results in a table, like this:

Name of object	Measure of circle c	Measure of diameter d	c-d	$\frac{c}{d}$
Water glass	____	____	____	____

   (d) To compare two quantities, we find their difference, or find their ratio. Find the difference between the circumference and diameter of each of your circular objects, and write the results in the third column, headed "c - d". Do the differences seem to be the same?

   (e) Find the ratios, by dividing the measure of the length of each circle by the measure of its diameter. Write the quotient, to the nearest tenth, in the fourth column of your table. Do the ratios seem to be about the same, for all three circles?

(f) The results in the fourth column should suggest that the circumference of each circle is about ___?___ times as long as its diameter.

(g) Compare the ratios in your $\frac{c}{d}$ column with those of your classmates. What seems to be true about them?

2. (Method II.)

(a) On stiff paper or cardboard mark a point P and draw a circle with center P. Make the diameter of the circle between 2 inches and $2\frac{1}{2}$ inches. Then cut along the circle so that you have a circular figure with the circle you drew as its boundary.

(b) Draw a line ℓ about 10 inches long, and label a point near the left end of the line point A. Then locate point B on ℓ so that the segment $\overline{AB}$ has the same length as the diameter of your circle.

(c) Lay out a number scale on line ℓ, with 0 at point A and 1 at point B. Extend the number scale to the number 4 or 5.

(d) Now mark a point on your circle and name it point C. Place the circle so it is tangent to your number line, with C at the zero point.

(e) Carefully roll the circle to the right along the number line. Each point on the circle will touch a point on the line. Continue rolling the circle until point C is again on the line. Mark the point where C touches the line point D.

(f) Between what whole numbers is point D? Estimate, to the nearest tenth, the decimal number which corresponds to point D.

(g) When you rolled circle P along the number line, each point on the circle touched exactly one point of the line, and no two points of the circle touched the same point of the line. What segment on the number line has the same length as the circle?

(h) What segment on the number line has the same length as the diameter of the circle?

(i) How does the length of the circle seem to compare with the length of its diameter?

(j) Does this result agree with your answer to Problem 1(f)?

Do the results of your experimentation in Problems 1 and 2 suggest the following statements?

1. For any circle, the ratio of the length of the circle to the length of its diameter is always the same number.
2. This number is a little greater than 3.
3. If experiments are carefully done, results suggest that this number is between 3.1 and 3.2.

Mathematicians have proved that the first conclusion above is correct. A special symbol is used for the number which is the ratio of the length of a circle to the length of its diameter. This symbol is written "π" and is read "pi". (It is a letter from the Greek alphabet, and is the first letter in the Greek word for "perimeter.")

We may state the relation of the measure c of a circle to the measure d of its diameter as: $\frac{c}{d} = \pi$ or $c = \pi \cdot d$.

Suppose you know the length of a circle and wish to find the length of its diameter. Since c is the product of the factors π and d, $\frac{c}{\pi} = d$

What is the relation of the length of a circle to the length of its radius?

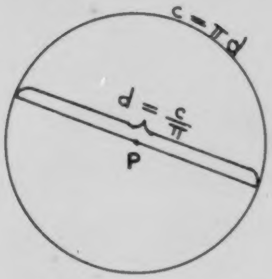
$$\text{Since} \quad d = 2 \cdot r, \quad \text{and} \quad c = \pi \cdot d$$

$$\text{then} \quad c = \pi \cdot (2 \cdot r)$$

$$\text{or} \quad c = (\pi . 2) \cdot r$$

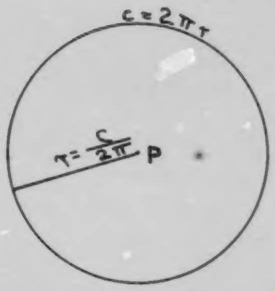
$$\text{or} \quad c = (2 \cdot \pi) \cdot r$$

And since c is the product of the two factors (2π) and r,



$$\frac{c}{2\pi} = r$$

$$\text{and } \frac{c}{r} = 2\pi$$



The Number π

The number which the symbol "π" represents is a new kind of number. It is clearly not a whole number. Neither is it a rational number. Recall that any decimal expansion of a rational number is a repeating decimal expansion. Mathematicians have proved that π cannot be a repeating decimal. A decimal expression for π to 10,000 decimal places was published in the year 1958.

Here is the decimal for π to fifty-five places.

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 36510 58209...

(The three dots at the end indicate that the decimal expression continues indefinitely.)

If you examine this decimal for the number π you see that it is difficult to locate the point on the number line to which it corresponds. However, by examining the digits in order we can find smaller and smaller segments on the number line which contain the point for π.

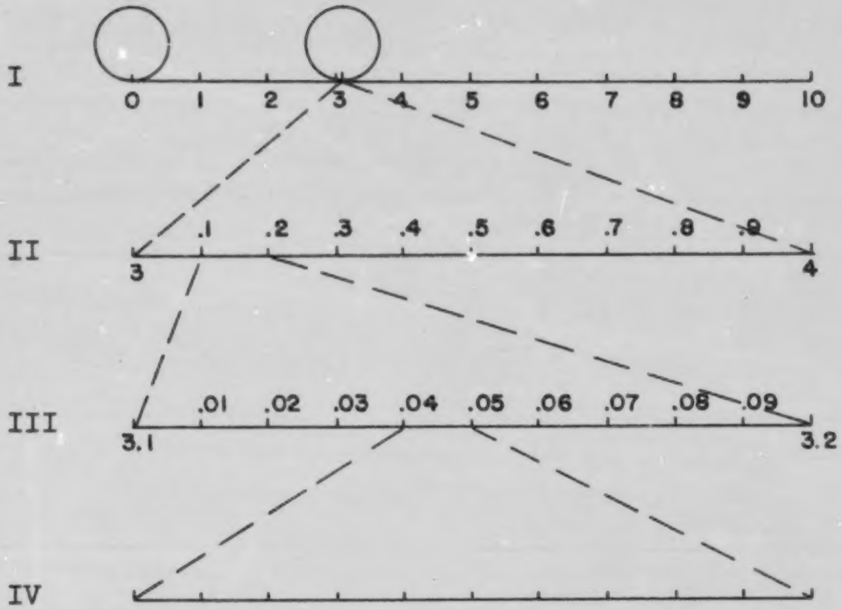
Study the following statements:

1. $\pi = 3 + .141592\ldots$ Therefore $\pi > 3$, and $\pi < 4$.
   Why? $(3 < \pi < 4.)$
2. $\pi = 3.1 + .041592\ldots$ Therefore $\pi > 3.1$, and $\pi < 3.2$.
   (Why?) $(3.1 < \pi < 3.2)$
3. $\pi = 3.14 + .001592\ldots$ Therefore $\pi > 3.14$, and $\pi < 3.15$.
   (Why?) $(3.14 < \pi < 3.15)$

What should the next statement be?

In the figure 10-5-a is shown a number line such as you used in Problem 2, Exercises 10-5a. The circle with diameter one unit in length is shown in its first position at point 0, and also in its position after rolling along the number line. The approximate position of the point which corresponds to the number π is the point of tangency, C.

Line I in the figure illustrates Statement 1 above. The point corresponding to π is on the segment with endpoints 3 and 4.



The segment on Line I with endpoints 3 and 4 is shown enlarged (ten times as large) on Line II. The segment is subdivided to show tenths, so the points of division correspond to the numbers 3.1, 3.2, 3.3, etc. Statement 2 tells us that the point for π is on the segment with endpoints 3.1 and 3.2.

The segment with endpoints 3.1 and 3.2 is shown enlarged (ten times as large) on Line III. The points on Line III subdivide it into tenths, and correspond to the numbers 3.11, 3.12, etc. From Statement 3 we know that the point corresponding to π is on the segment with endpoints 3.14 and 3.15.

What numbers should be at the endpoints of the segment marked IV? How should the points of subdivision be labeled? On what segment of line IV is the point for π?

Look again at line III. Notice that the whole segment on line III was made 100 times as large as it would actually be on the number line I, on which the circle was rolled. Using just three digits in the decimal for π we have shown that the point π lies on a particular segment which is very small - just $\frac{1}{100}$ of the segment of the number line between the points 3 and 4.

If you used one more digit, you could show that the endpoints of a segment which contains π are _?_ and _?_.

Exercises 10-5b

In Problems 1 and 3-6, use $\pi \approx 3.14$. In Problems 7-8, use $\pi \approx 3.1$.

1. Find the missing information about the circles described.
   (a) Circle C: Radius, 4.2 cm. Diameter, _?_ Circumference _?_
   (b) Circle D: Radius, _?_ Diameter, 5.6 in. Circumference _?_
   (c) Circle E: Radius, _?_ Diameter, _?_ Circumference, 25 ft
2. (a) Find, to three decimal places, the difference between π and each of the following rational numbers:

   $\frac{19}{6}$ $\frac{22}{7}$ $\frac{25}{8}$

   (b) Which of the rational numbers in problem 2 is nearest π?
   (c) Using $\pi \approx \frac{22}{7}$, find the following:
      (1) The length of a circle whose diameter is 14 in.
      (2) The length of a circle with radius 21 ft.
      (3) The diameter of a circle with circumference 132 in.
      (4) The radius of a circle with circumference 44 ft.
      (5) The circumference of a circle with radius $10\frac{1}{2}$ in.
3. A circular lampshade 12 in. in diameter needs new binding around the bottom. How long a strip of binding will be needed? (Disregard any needed for overlap.)

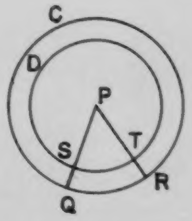
4. A strip of metal 62 inches long is to be made into a circular hoop. What will its diameter be? Would it be large enough for the hoop for a basketball basket? (Official diameter is 18 inches.)

5. A merry-go-round in a playground has a 15-foot radius. If you sit on the edge, how far do you ride when it turns once?

6. A wheel moves a distance of 12 feet along a track when the wheel turns once. What is the diameter of the wheel?

7. A circle with a diameter of 20 inches is separated by points into 8 arcs of equal length.
   (a) What is the length of the whole circle?
   (b) What is the length of each arc?
   (c) What is the arc measure of each arc?
   (d) On this circle, how long is an arc of one degree?

8. In the figure, circle C and circle D have the same center P. The radius of circle C is 7 in. and the radius of circle D is 5 in.



   (a) Find the length of each circle.
   (b) If angle QPR contains 70 degrees, what is the arc measure of arc $\widehat{ST}$? of arc $\widehat{QR}$?
   (c) What fractional part of circle C is arc $\widehat{QR}$? What fractional part of circle D is $\widehat{ST}$?
   (d) What is the linear measure of arc $\widehat{QR}$? of arc $\widehat{ST}$?

9. (a) Find, to four decimal places, the decimal for 2π.
   (b) Find, to five decimal places, the decimal for 3π.

10. Sometimes it is a good idea to use π as a numeral, instead of using a decimal for π. Answer the following questions using π as a numeral. We say the answer is expressed in terms of π.

    (a) If the length of a circle is 54π in., what is its diameter? its radius?

    (b) If the diameter of a circle is 13 in., what is the length of the circle?

    (c) If the radius of a circle is 3.6 cm., what is the length of the circle?

*11. Suppose that the diameter of circle C is three times as long as the diameter of circle D. What is the ratio of their circumferences? (Hint: Think of lengths for the diameters, and then find the circumferences. Use π as a numeral.)

12. BRAINBUSTER. Suppose a tight band is placed around the earth, over the equator. Then suppose the band is made 1 foot longer which loosens the band, and that the band is the same distance from the earth all the way around.

    (a) Could a mouse crawl under the band?

    (b) Could you crawl under the band? If not, how much longer would the band have to be to make it possible for you to crawl under it?

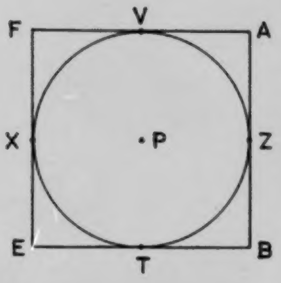
    (c) What mathematical property could you use to answer these questions?

10-6. Area of a circle

In the kitchen at your house you may find a circular frying-pan called a nine-inch skillet. The boundary of the frying surface represents a circle, and "nine-inch" tells the diameter of the circle. Some electric skillets are square-shaped. Perhaps in your kitchen there is an eight-inch square frying-pan. If you look at a nine-inch circular pan and at an eight-inch square pan, you may ask which is bigger. By "bigger" we mean more surface usuable for frying. In other words, we may want to compare the area of the interior of the circle with the area of the interior of the square. After a careful inspection of the two skillets, you likely would conclude that the areas are so close that you cannot decide by appearance which is greater.

Since each side of the square is eight inches long, we know the area of the interior is sixty-four square inches. But we have not yet studied a method of computing the area of the interior of a circle whose diameter is known.

When speaking about a circle in everyday language, we customarily abbreviate the phrase "the area of the interior of the circle" and simply say "the area of the circle". Very often the area of a circle is expressed in terms of the radius of the same circle.



In the figure, the point P is the center of the circle and the center of the square ABEF. Let the measure of a radius of the circle be r. Each of the segments $\overline{XP}$ and $\overline{PZ}$ is a radius. Explain why the measure of the diameter $\overline{XZ}$ is 2r. What is the measure of the segment $\overline{EB}$? What is the measure of the segment $\overline{AB}$? Explain your reply. What is the measure of the interior of the square ABEF? Does the interior of ABEF have four times as much area as the interior of square AZPV?

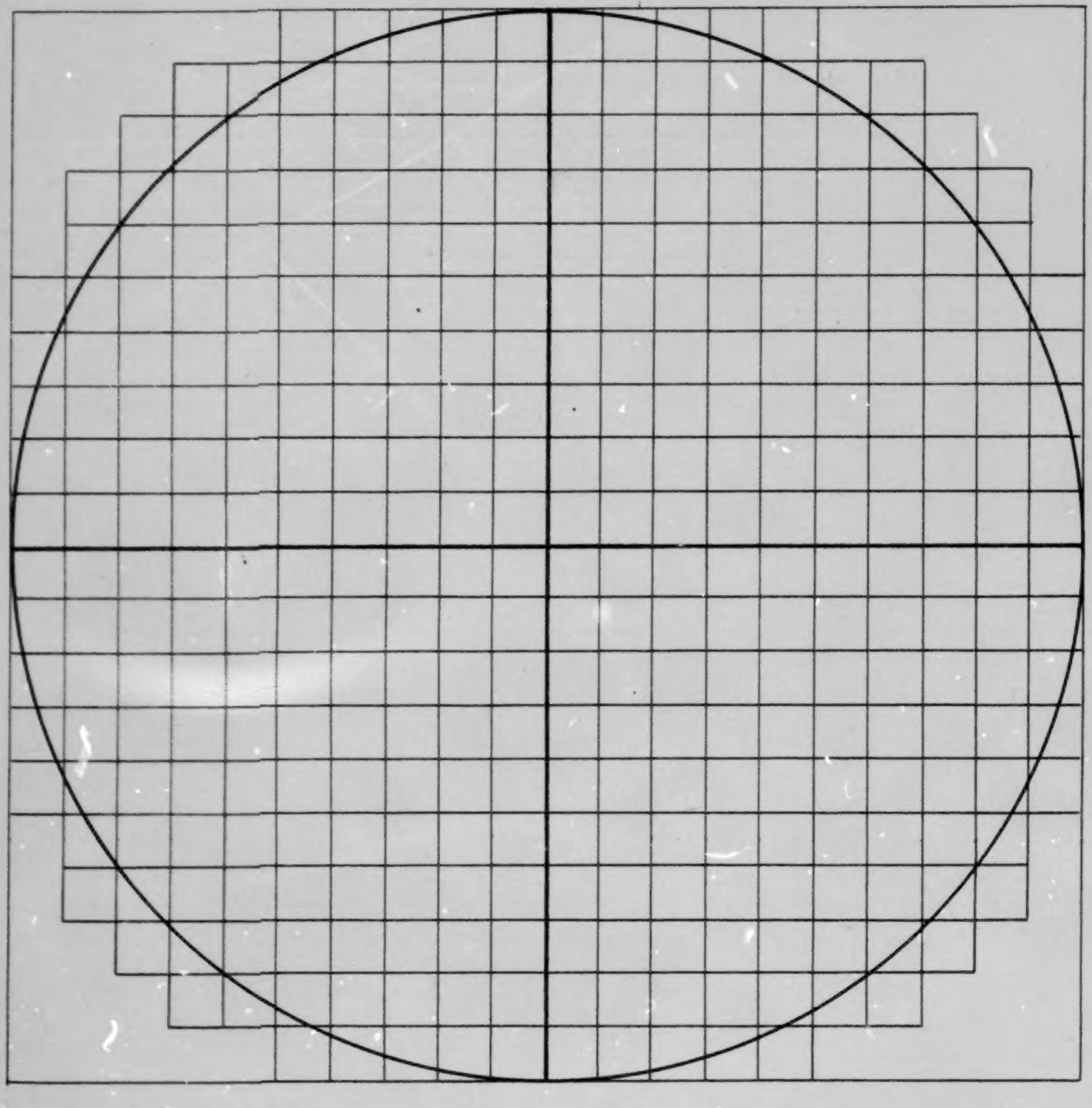
How does the area of the interior of the circle compare with the area of the interior of ABEF? Do you believe that a circle whose radius is r units has an area which is less than $4 \cdot r^2$ units?

After a careful inspection of the figure do you estimate that the measure of the interior of the circle is more than one-half the measure of the interior of the square? More than two-thirds? More than three-fourths?

Now let us try approximating the area of a circle by a rather careful measurement. You may wish to review in Chapter 7 the basic notion of a unit of measure and how we estimate a size by finding how many units are contained in the figure.

On the next page, at the top, is shown a square, each of whose sides is one unit long. The large figure shows a circle whose radius is ten units in length. The region has been covered with units of area. Count how many of the units of area do not meet the exterior of the circle at all.

Several of the units of area have the property that an arc of the circle separates them into two parts. In some cases the larger part is inside the circle. Mark each square with this property, and then count how many there are. Inspect the squares for which the smaller part is inside the circle. Is the total area of these small pieces inside the circle an approximation to the total area of the portion of the marked squares outside the circle? Do you believe that a good way to estimate the area of the portion of the region near the circle itself is to disregard these small pieces and to count the marked squares in full (as if they were completely inside the circle)? Now that you have finished your counting, what

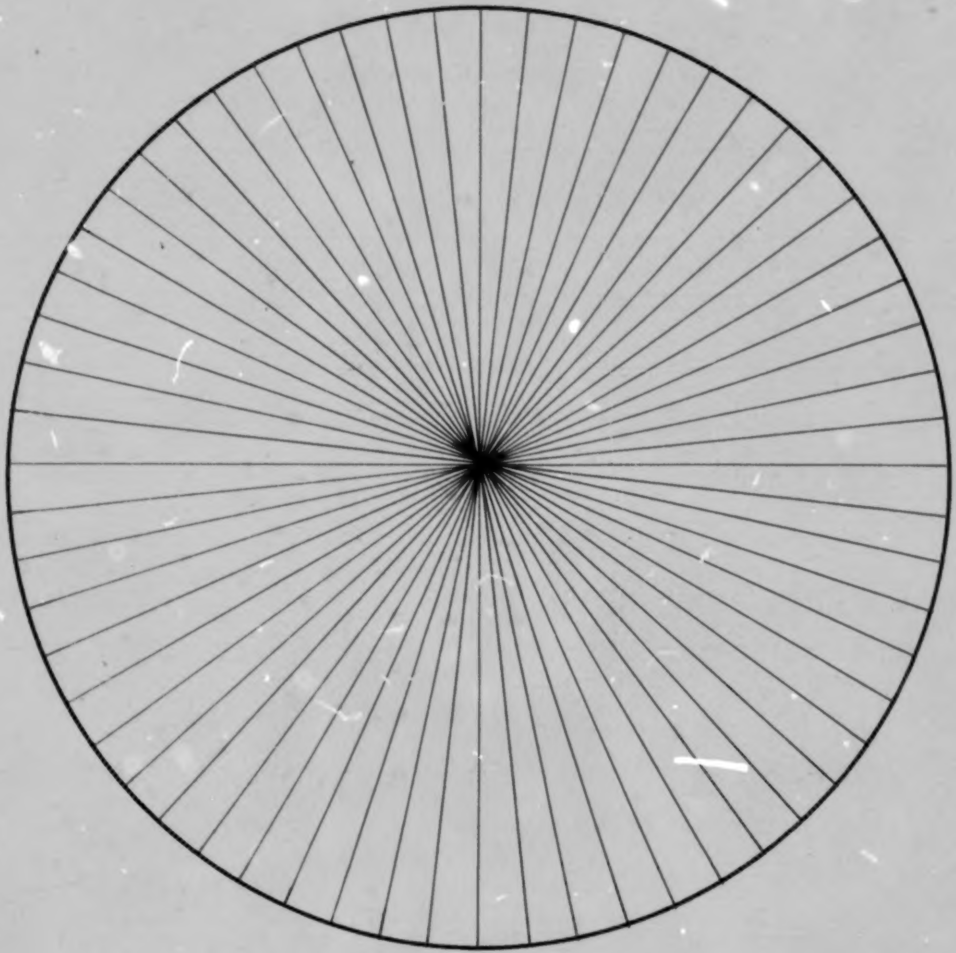


2 ,1

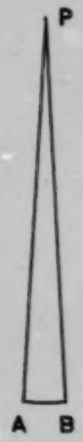
measure, in terms of the given unit of area, do you think should be given to the interior of the circle? Compare your response with that of your classmates.

Now take a sheet of rectangular graph paper, and on it draw (with your compass) a large circle. Using the shortest unit of length printed on the paper, measure the radius of the circle. Count the number of units of area which the interior of the circle has. As before, when a unit square is separated by an arc of the circle, count it if you estimate that more than half of the interior of the square is inside the circle. But do not count it if you believe that the greater portion is outside the circle. Your observed measure of the area is how many times the second power of the measure of the radius?

We have obtained by measurement an estimate of the area of certain circles. Now we will adopt a different approach toward the problem of determining the area. Let us consider a circle and suppose that the number r is the measure of a radius. Suppose that many radii of the circle are drawn in a regular pattern. The diagram now resembles a wheel with many spokes. In Figure 10-6 we chose to have sixty radii. In order to have a definite number to speak about in the remainder of the discussion, we will continue to use this choice 60. The sixty rays separate the circle into sixty arcs, all of the same length. They also separate the interior of the circle into sixty regions, all of the same area.



We fix our attention on a typical one of these regions, as shown in the figure. This region appears to be nearly triangular; indeed, only a short portion of its boundary, namely the arc AB, is not straight. Perhaps we can approximate the area of this region by finding the area of the interior of a triangle. (Do you recall the method for computing the area of a triangular region?) We choose the triangle whose base is as long as the arc AB and whose altitude is the same as the distance of P from the arc.



Since the length of the entire circle is $2\pi \cdot r$, the length of the arc is $\frac{1}{60} \cdot 2\pi r$.

The distance between P and the arc is simply the radius of the circle.

Thus for the approximating triangle, as described, the area of its interior is $\frac{1}{2}\left(\frac{1}{60} \cdot 2\pi r\right)(r) = \frac{1}{60} \cdot \pi \cdot r \cdot r$.

An approximation to the area of the region bounded by the curve ABP is $\frac{1}{60} \cdot \pi r^2$. Since there are sixty such regions inside the circle (Figure 10-6), an estimate of the area of the circle is 60 times $\frac{1}{60} \cdot \pi r^2$, that is, πr^2.

How does this approximation compare with the results obtained by actual measurements which you made earlier?

In reality the interior of a circle with radius r units of length has exactly πr^2 units of area. Thus, although our method of development used an approximating step, our result agrees with the correct result that may be firmly established in more advanced mathematics.

As an application, let us return to the comparison of the frying-pans. The nine-inch skillet has a radius of $\frac{9}{2}$ inches. The number of square-inches in its area then is $\pi\left(\frac{9}{2}\right)^2 = \frac{81\pi}{4}$, which is nearly 63.6. (What decimal approximation to π was conveniently used in this case?) In conclusion, which has the greater frying surface, the nine-inch circular skillet or the eight-inch square frying-pan? Is there enough difference between them to be of significance to the cook?

Exercises 10-6

1. Information is given for four circles. The letters r, d, c, A are the measures of the radius, the diameter, the length (or circumference), and the area respectively. Find all the missing data. Use 3.1 as an approximation to π.

r	d	c	A
20	____	____	1240
____	8	____	____
____	____	0.093	____
____	222	____	____

Note that $\pi(20)^2 \approx (3.1)(400) = 1240$

2. Which has the greater frying surface--an eight-inch circular skillet or a seven-inch square frying-pan?

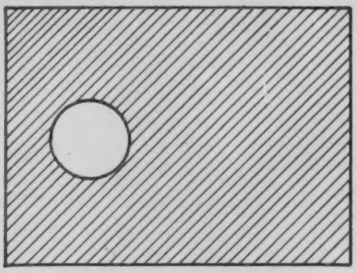
3. A circular drum-head is twelve inches across. What is the area of the drum-head?

4. Which method is easier for finding the area of a circle--to measure the radius and calculate the area, with the aid of πr^2, or to measure the area directly with an appropriate unit of

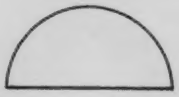
measure? Should both methods yield the same result?

5. The earth is about 150 million kilometers from the sun. The orbit (or path) of the earth around the sun is not really circular, but approximately so. Suppose that the orbit were a circle; then the path would lie in a plane and there would be an interior of the orbit (in the plane). What would be an estimate for the area of this interior?

6. A rectangular plot of land, 40 feet by 30 feet, is mostly lawn. The circular flower-bed has a radius of 7 feet. What is the area of the portion of the plot that is planted in grass?



7. The figure represents a simple closed curve composed of an arc of a circle and a diameter of the circle. The area of the interior of this simple closed curve, measured in square inches, is 8π. Do not use any approximation for π in this problem.



(a) What is the area of the interior of the entire circle?

(b) What is the second power of the radius of the circle?

(c) How long is a radius of the circle?

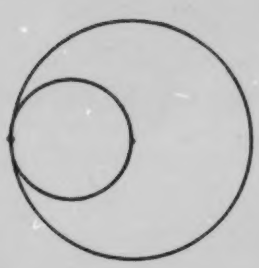
(d) How long is the straight portion of the closed curve represented in the figure?

(e) What is the circumference of the (entire) circle?

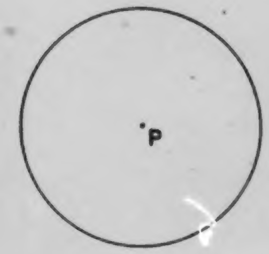
(f) How long is the circular arc represented in the figure?

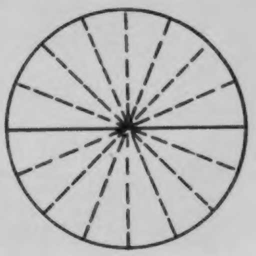
(g) What is the total length of the simple closed curve?

8. The center of the longer circle lies on the shorter circle. The intersection of the two circles is a single point. This point and the centers of the two circles lie on one lie, If the interior of the shorter circle is chosen as a unit of measure, what would be the measure of the region inside the longer circle and outside the shorter circle?

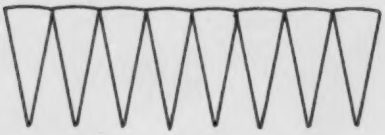


9. Another way to discover a relationship between the radius of a circle and the area of the same circle is the following. Perform the suggested steps. On a piece of stiff paper draw a large circle with your compass. Mark the center P of the circle. (See left-hand figure.) Use your protractor to assist you in drawing eight lines on P which will separate the interior of the circle into sixteen regions all of the same area. (See right-hand figure.) Can you calculate how many arc-degrees each of the sixteen arcs will have?

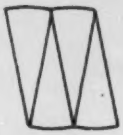




Cut away the portion of the paper outside the circle. Cut through the interior of the circle along the fully-drawn diameter. In each of the two halves, cut with great care along the dotted radii from P almost to the circle itself. The eight angular portions should hang together somewhat like teeth.



With both portions cut in a toothed fashion, fit the two pieces together. (Only a few of the sixteen teeth are shown in this figure.)



The upper and lower boundaries of the completed pattern have a scalloped appearance. If they were straight, the entire figure would be the interior of a __________(fill the blank with the best choice of a name for this simple closed curve.) As an application of results from Chapter 9 you may estimate the area of the interior of this curve by using the measures of its apparent base and its apparent altitude. How does your result compare with the product of π and the second power of the radius?

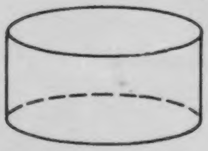
10. **BRAINBUSTER.** A barn is 40 feet long and 20 feet wide. A chain 35 feet long is attached to the barn at the middle of one of its longer sides. Another chain 35 feet long is attached to the barn at one of its corners. Either chain may be used to tether a cow for grazing.

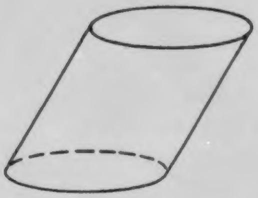
    (a) Which chain gives the tethered cow the greater area of land over which to graze?

    (b) How much difference is there between the areas of the two regions? (Use 3.1416 as an approximation to π.)

10-7. Cylindrical Solids

In Chapter 7, you studied the rectangular solid, its volume and its surface area. In Chapter 9, you studied the prism, its volume, and its surface area. Here we shall study another solid which is frequently found in everyday life. Instead of having a rectangular region as a base, like a box, suppose a solid has a circular region as a base, like a tin can. We call such a solid a cylindrical solid (or sometimes just a cylinder). You are familiar with other examples of cylindrical solids such as water pipes, tanks, silos, and some drinking glasses.





In the figures above, the left hand and middle drawings represent "right" cylinders and the right hand drawing represents a "stanted" or "oblique" cylinder. We rarely deal with slanted cylindrical solids in ordinary life. Therefore in this chapter we shall assume our solids are "right" cylindrical solids.

We list some important properties of a (right) cylinder.

(1) It has two bases (a top and a bottom) and each is a circular region. Their areas are equal.

(2) Each base is in a plane and the planes of the two bases are parallel.

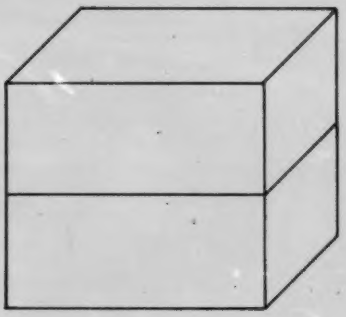
(3) If the planes of the bases are regarded as horizontal, then the upper base is directly above the lower base.

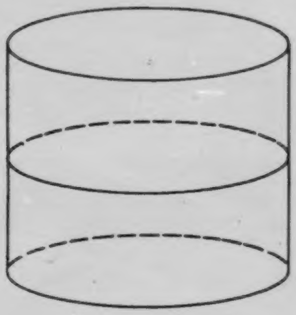
(4) The lateral or side surface of the cylinder is made up of the points of segments each joining a point of the lower circle with the point directly above it in the upper circle.

There are two numbers or lengths which describe a cylindrical solid. These are the radius of the base of the cylinder and the altitude (or height) of the cylinder. The altitude is the (perpendicular) distance between the parallel planes containing the bases. For a right cylinder, the altitude can also be thought of as the length of one of the segments lying in the lateral surface and joining the two bases.

We now ask ourselves about methods of finding the volume of a cylindrical solid. In one sense there is a fairly easy method. If the solid is like a tin can and will hold water (or sand) we can fill it up and then pour it into a standard container. We would like to know what the answer is without having to do this every time and for some cylinders this would be very impractical.

Recall how we found the volume of a box or of a (right) prism. We first considered a box or prism one unit high. The number of cubic units in this box or prism would be the same as the number of square units in the base. Thus the measure of the volume was clearly the measure of the area of the base times one. If the box or prism had an altitude of two units, then the measure of the volume would clearly be twice as much as the measure of the area of the base. That is, it would be 2 times the measure of the area of the base.





In general, if the area of the base were B square units and the altitude of the box or prism were h units then the volume would be B·h cubic units.

Exactly the same situation occurs with a cylindrical solid. The measure of the volume of the cylinder is simply the measure of the area of the base times the measure of the altitude. The area of the base of a cylinder is πr^2 square units. So the volume is $\pi r^2 . h$ cubic units.

We now have one basic principle which applies to boxes, to other prisms, and to cylinders. The measure of the volume is the measure of the area of the base times the measure of the altitude.

You should learn and remember how to compute the volume of a cylindrical solid. For most of you it is probably better not to memorize the formula $\pi r^2 h$ as such. To compute the volume of any solid of this type we simply multiply the measure of the area of the base by the measure of the altitude. And the altitude is simply the (perpendicular) distance between the parallel planes which contain the bases. If you think of the geometrical figure and what it is you want to find, then most problems of this type are very easy.

A note on computation. When making computations involving π, it is usually easier to use a decimal approximation for π only at the last step of the arithmetic. In this way we use long decimals as little as possible. Consider $\pi \cdot 5^2 \cdot 8$. We note that $5^2 = 25$ and that $25 \cdot 8 = 200$. Therefore $\pi \cdot 5^2 \cdot 8 = \pi \cdot 200 \approx (3.14) \cdot (200) = 628$. This procedure is much simpler than multiplying 3.14 by 25 and then multiplying the result by 8.

Exercises 10-7

1. A silo (with a flat top) is 30 feet high and the inside radius is 6 feet. How many cubic feet of grain will it hold? (What is its volume?) Use $\pi \approx 3.14$.
2. A cylindrical water tank is 8 feet high. The diameter (not the radius) of its base is 1 foot. Find the volume (in cubic feet) of water which it can hold. Leave your answer in terms of π. If you use an approximation for π, what is your answer to the nearest (whole) cubic foot?
3. There are about $7\frac{1}{2}$ gallons in a cubic foot of water. About how many gallons will the tank of Problem 2 hold?

4. Find the amount of water (volume in cubic inches) which a 100 foot length of pipe will hold if the inside radius of a cross-section is 1 inch. Use $\pi \approx 3.14$. (A cross-section is shaped like the base. A cross-section is the intersection of the solid and of a plane parallel to the planes of the bases and between them.)

5. Find the volume of a cylindrical solid whose altitude is 10 centimeters and the radius of whose base is 3 centimeters. Leave your answer in terms of π.

6. In the problem of (5), what would the volume be if the altitude were doubled (and the base were left unchanged)?

7. In the problem of (5), what would be the volume if the radius of the base were doubled (and the altitude were left unchanged)?

8. In the problem of (5), what would be the volume if the altitude were doubled and the radius of the base were also doubled? (Think of first doubling the altitude and then doubling the radius of this new cylindrical solid.)

*9. In general, what is the effect on the volume of a cylindrical solid obtained by doubling the altitude? By doubling the radius of the base? By doubling both the doubling both the radius of the base and the altitude?

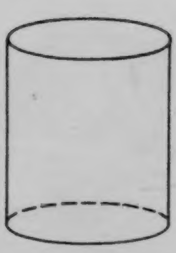
10. BRAINBUSTER. What is the amount (volume) of metal in a piece of water pipe 30" long if the inside diameter of a cross-section is 2" and the outside is 2.5". Use $\pi \approx 3.1$.

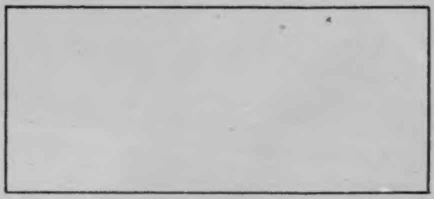
10-8. Cylindrical Solids--Surface Area

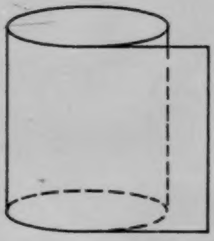
In the previous section we have considered questions about the volume of a cylinder. Now we ask about the surface area.

There are two questions we might ask. What is the surface area of the curved part (the lateral area)? What is the total surface area? It is easy to see the relation between these two if we think of the solid itself. The total area is the lateral area plus the area of the top base plus the area of the bottom base. But the areas of the top and bottom bases are the same. And the measure of each is πr^2 where r is the measure of the radius of the base. So if we knew how to find the lateral surface area, we would also know how to find the total area.

The label of a tin can covers the curved surface of the can. The area of the label is the lateral area of the cylinder. How are labels made? They are made and printed in the form of rectangular regions. The height of such a rectangle is the height of the cylindrical solid. The length of the base is the length of the base circle of the cylinder. (or, when made, it is this length plus a little more to allow for overlapping.) The lateral area of a cylinder, then, is the area of a rectangle which will just cover it.







We have observed that the lateral area of a cylinder is the area of a certain rectangle. The altitude of the rectangle and the altitude of the cylinder are the same. The length of the base of the rectangle and the length of the base circle of the cylinder are the same. Therefore the measure of the lateral area of the cylinder is the product of the measure of the length of the base circle and the measure of the height. Hence, the measure is $2\pi r \cdot h$. And the measure of the total area is then $2\pi r \cdot h + 2\pi r^2$.

There are some curved surfaces, like the surface of a ball, which cannot be treated in quite this simple a way. Rectangular regions, or other flat surfaces, just don't wrap nicely around them. The areas of such surfaces can be handled in other ways. It is fortunate that cylinders have "easy" curved surfaces.

Which new formulas of this chapter should you learn? There are two of vital importance. You should know these by heart and without fail.

(1) The measure of the length of a circle is $2\pi r$ (or πd).

(2) The measure of the area of a circular region is πr^2.

The formulas for the measures of the volume and surface area of a cylindrical solid need not be learned, as such, if you are able to think of what the formulas represent.

You should understand and know the basic general principle for getting the volumes of many solids. For boxes, for other prisms, and for cylinders this principle tells us that the measure of the volume is the measure of the area of the base times the measure of the altitude.

Finally, in many problems such as surface area you should think of the geometric objects and what it is about these that you want to know. As an example, the lateral surface represents (if cylindrical solid is obtained by thinking of what the lateral surface represents (if flattened out). Then we get the measure of the lateral area as the measure of the length of the base circle times the measure of the height of the solid. For the measure of the total surface area we need only add to this measure twice the measure of the area of the base.

Exercises 10-8

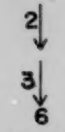
1. Take an ordinary tin can. Measure it and make a label for it that will fit without overlapping. Your label will be in the form of a __________. Its height will be the __________ of the tin can. The length of its base will be the length of the ________ __________ of the tin can. Actually try your label to see that it fits.
2. Take another tin can of different size than that of Problem 1. Make a label for it without doing any measuring. You should determine the desired size of the label by comparison with the tin can.
3. With a tape measure or string (to be measured) figure out the lateral surface area of the tin can of Problem 2 by measuring the length of the base circle and the height directly.
4. Measure the diameter (and then figure out the radius) of the base circle of the tin can of Problem 2. Using this and the height find the lateral surface area of the tin can. Should your answer agree with the answer you got in Problem 3?
5. Find the lateral surface area (in square centimeters) of a cylindrical solid whose altitude is 8 centimeters and the radius of whose base circle is $1\frac{1}{2}$ centimeters. Use $\pi \approx 3.14$.
6. Find the total surface area of the cylinder of Problem 5.

7. How many square meters of sheet metal do you need to make a closed cylindrical tank whose height is 1.2 meters and the radius of whose base circle is .8 meters? How many square meters do you need if the tank is to be open on top?

8. A small town had a large cylindrical water tank that needed painting. A gallon of paint covers about 400 square feet. How much paint is needed to cover the whole tank if the radius of the base is 8 feet and the height of the tank is 20 feet? Give your answer to the nearest tneth of a gallon.

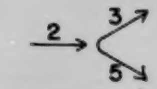
Exercises 10-9. Review of Chapters 9 and 10

1. It is often helpful to use diagrams to show relations. For example



may represent $2 \cdot 3 = 6$

Relations which involve sums as well as products may be shown by diagrams.



may represent

$$2\,(3 + 5) = 2 \cdot 3 + 2 \cdot 5$$
$$= 6 + 10 .$$

You have studied ways of computing perimeters, areas, and volumes for a number of geometric figures and their interiors; and you have expressed these methods briefly in number sentences or formulas.

Below are twelve diagrams for several of these formulas, and a list of the geometric figures to which one (or more) of the formulas applies. Copy each diagram on your paper, and beside it write the following:

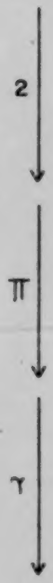
(a) The name of the figure (or figures) to which the diagram applies. (Choose from the list of figures below.)

(b) What the formula tells about the figure: area, perimeter, volume, circumference, lateral area.

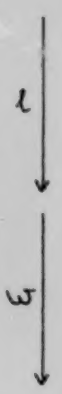
(c) The formula shown in the diagram.

List of figures: parallelogram
rectangular prism
circle
triangle
rectangle
triangular prism
cylinder

(1)



(2)



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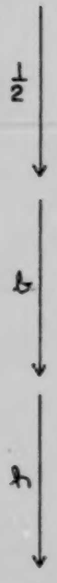

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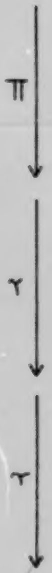
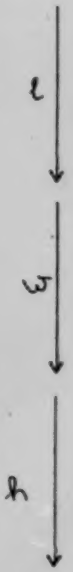


(5)



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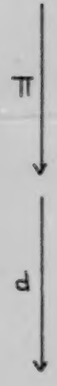

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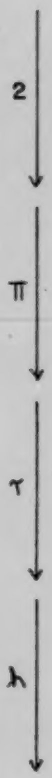

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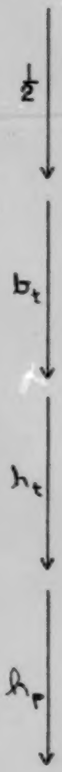
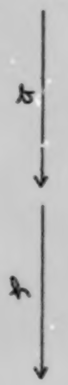
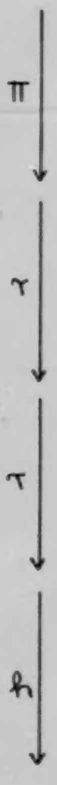


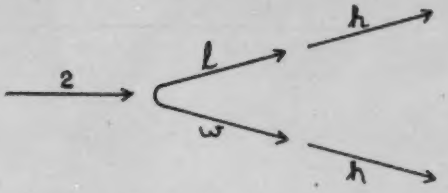
(9)



(10)



(11)



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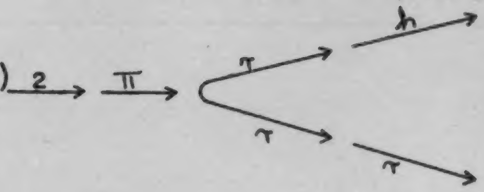


2. (a) In your copies of the diagrams, draw a circle around each numeral.

   (b) Draw a square around each symbol for a measure you would have to know in order to apply the formula to a particular figure.

   (c) Hold a ruler horizontally across the diagrams (1) to (10) in your text, to separate the numerals from the symbols for measures. Why are there more arrows below your ruler in some diagrams than in others?

   (d) Each of the formulas tells how to find the measure of a quantity which has either one dimension, two dimensions, or three dimensions. Put a "1", "2", or "3" beside each diagram, to show the dimension of the kind of quantity to which it relates.

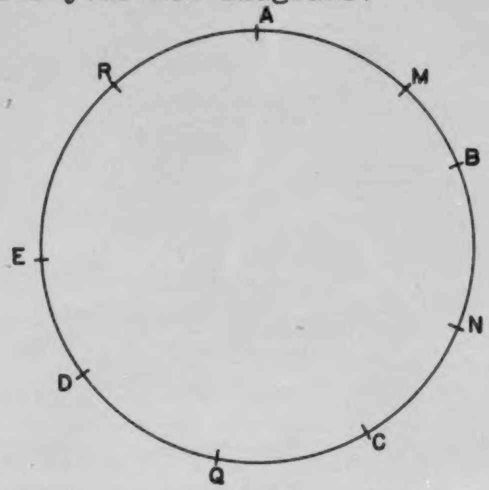
   (e) Do you see any connection between your answers for Questions 2(b) and 2(d)?

   (f) Look at the diagrams of formulas for finding volumes. (There are three.) Put a B at the end of one arrow, to separate the part of the formula which gives the area of the base from the rest of the formula.

*(g) Make up formulas relating to squares and cubes. Make diagrams for the formulas, and tell what each diagram means.

*(h) Draw the diagrams 11 and 12 in a different way. Write the formulas for your new diagrams.

3.



In the above figure determine with the use of your protractor the measure of the following arcs. Indicate your results with correct use of symbols.

(a) $\overset{\frown}{AMB}$

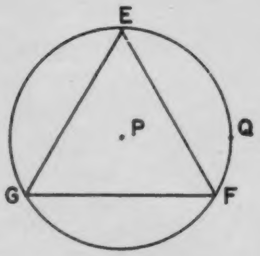
(b) $\overset{\frown}{ABC}$

(c) $\overset{\frown}{CQD}$

(d) $\overset{\frown}{CDE}$

(e) $\overset{\frown}{RAM}$

4. Each side of triangle EFG is measured as 34.6 meters long. The distance between E and the center P of the circle is measured as 20.0 meters. The altitude of triangle EFG from E to the side $\overline{FG}$ is



measured as 30.0 meters long.

(a) What is the area of the interior of triangle EFG.

(b) What is the area of the interior of the circle? (Use 3.142 as an approximation to π, and round your answer to the nearest square-meter.)

(c) On your paper draw a sketch of the figure and shade the intersection of the interior of the circle and the exterior of the triangle EFG.

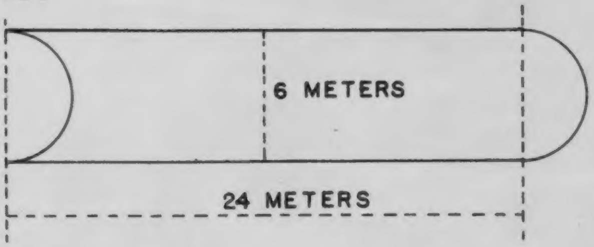
(d) What is the area of this intersection?

(e) The halfplane on the Q-side of EF and the interior of the circle have an intersection. What is the area of this intersection?

5. Below are four figures representing simple closed curves. Each curve is the union of several segments and one or two arcs of circles. Each arc either is a semi circle or is measured as 90 arc-degrees. The dotted segments are not parts of the simple closed curves but are useful in indicating the lengths.

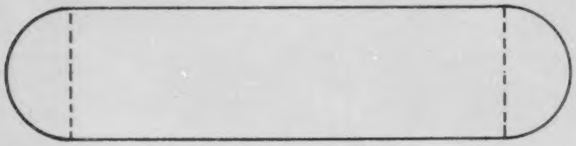
For each curve find its total length and find the area of its interior.

(a)



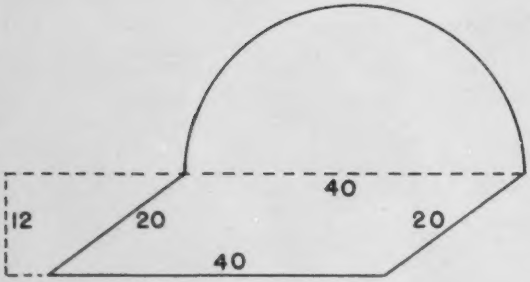
(Use $\pi \approx 3$.)

(b)



Each segment is 31.4 centimeters long. The distance between the parallel segments is 17.9 centimeters. (Use $\pi \approx 3.14$, and round final answers to nearest centimeter or nearest square-centimeter.)

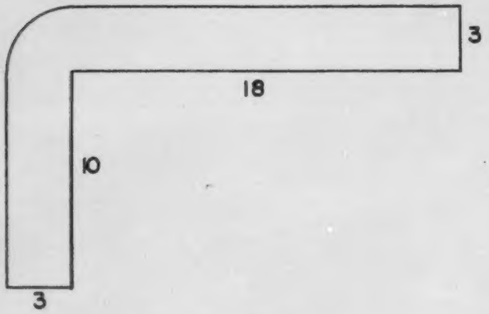
(c)



Unit of measure is foot.

(Leave your answer in terms of π.)

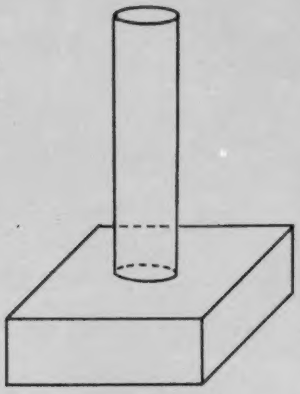
*(d)



Unit of measure is millimeter.

(Leave your answer in terms of π.)

6. Find the volume of a monument made as follows: The base is a rectangular block of marble of dimensions 4' by 6' by 2' high. On top of this block in the center is a cylindrical solid of height 8' and with the radius of the base circle 1'. Use $\pi \approx 3.1$.



*7. Find the total area of the exposed surface of the monument of the preceding problem. The underside of the base is considered as not exposed.

CHAPTER 11

STATISTICS AND GRAPHS

11-1. Gathering Data

If you look at the pupils in your classroom it may occur to you that several pupils appear to be about the same height, that some pupils are taller and some are shorter. Suppose you want to know the height of the tallest, the height of the shortest and how many are the same height. How will you go about finding out? You would first measure the height of each pupil. In this way you will be collecting facts to answer the questions you have in mind. Instead of collecting facts we may say you are gathering data. (The word "data" is the plural of the Latin word "datum" which means "fact".)

Let us suppose that you find each pupil's height correct to the nearest inch. After this is done, the measurements should be arranged in such a way that it is possible to select information from the data as easily as possible. Making such an arrangement is done frequently by putting the data in a table as shown in Table 11-1. If you wished, you might list the pupils by name but here each pupil is assigned a number.

With the data arranged in this way it is very easy to answer such questions as the following. How tall is the tallest pupil? the shortest? How many pupils are 60 inches tall or taller? How many are less than 60 inches tall? What height occurs most often?

Table 11-1

Heights of 15 Seventh Grade Pupils

Pupil	Heights in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	52

This little example about the heights of the pupils in a class is a sample of the kind of things we do in studying statistics. Statistics, in part at least, has to do with the collection of data and the making of tables and charts of numbers which represent the data. The tables and charts usually make it easier to understand the information which is contained in data that have been gathered. We will use the data in Table 11-1 later in this chapter to illustrate some other things that we do in our study of statistics.

Many of the duties of different agencies in the U. S. government could not be performed if the agencies were not able to collect a great many data to use in their work. The Congress of the United States has the power "to lay and collect taxes---to pay the debts and provide for the common defense and general welfare of the United States." The amount of taxes to be collected depends on many things. Name some of them. Certainly one thing on which it depends is the number of people in the United States. The Congress must provide for counting the people "within every term of ten years." The census taken in 1950 showed that there were about 151,000,000 people in the United States. When will the next census be taken?

Table 11-2 shows the population in millions for every census taken since 1790. The table shows that the population in 1790 was 3.9 millions. This means there were 3,900,000 (3.9 × 1,000,000) people in the U. S. at that time. Is this an exact or approximate number? The column which is headed Percent of Increase shows the percent of increase in the population during the preceding ten-year period.

Table 11-2

Population Facts About the United States*

Census Years	Population in Millions	Increase in Millions	Percent of Increase
1790	3.9		
1800	5.3	1.4	35.1
1810	7.2	1.9	36.4
1820	9.6	2.4	33.1
1830	12.9	3.3	33.5
1840	17.1	4.2	32.7
1850	23.2	6.1	35.9
1860	31.4	8.2	35.6
1870	39.8	8.4	26.6
1880	50.2	10.4	26.0
1890	62.9	12.7	25.5
1900	76.0	13.1	20.7
1910	92.0	16.0	21.0
1920	105.7	13.7	14.9
1930	122.8	17.1	16.1
1940	131.7	8.9	7.2
1950	150.7	19.0	14.5

*From Statistical Abstract of the United States, 1956.

Exercises 11-1

1. Do you see any general trends in the data shown in the table?
2. In which decade was the percent of increase the largest? Do you know a reason for this from your study of history?
3. In which decade was the percent of increase the lowest? Can you explain this by history you have studied?
4. The Irish Famine occurred in the years 1845, 1846, 1847. How did that affect the population of the United States?
5. What was the percent of increase in population from 1870 to 1880?

11-2. Broken Line Graphs

Such data as given in Table 11-2 are frequently represented by "drawing a picture". The picture for these data is shown in Figure 11-1. The two columns of data in the table which are headed Census Years and Population in Millions are used in drawing the "picture". In the "picture" in Figure 11-1 the census years 1790, 1800, ..., 1960 are shown along the horizontal base line. The distances between the points representing the census years are equal. Before deciding what the distance between the points should be, we first counted the number of census years (17 of them), then selected the distance between the points so that all points would be on the base line and far enough apart to make the "picture" easy to read. The Population column in Table 11-2 shows that the largest population to be represented is about 150 millions. Since 150 may be factored into 5 × 30 we decided to use 30 dots along the vertical line, each dot representing 5 million people. For this reason the vertical line at the left was made very long. The distance between the points on the vertical scale is small enough to get all the points on the page and large enough to make it easy to distinguish between the points.

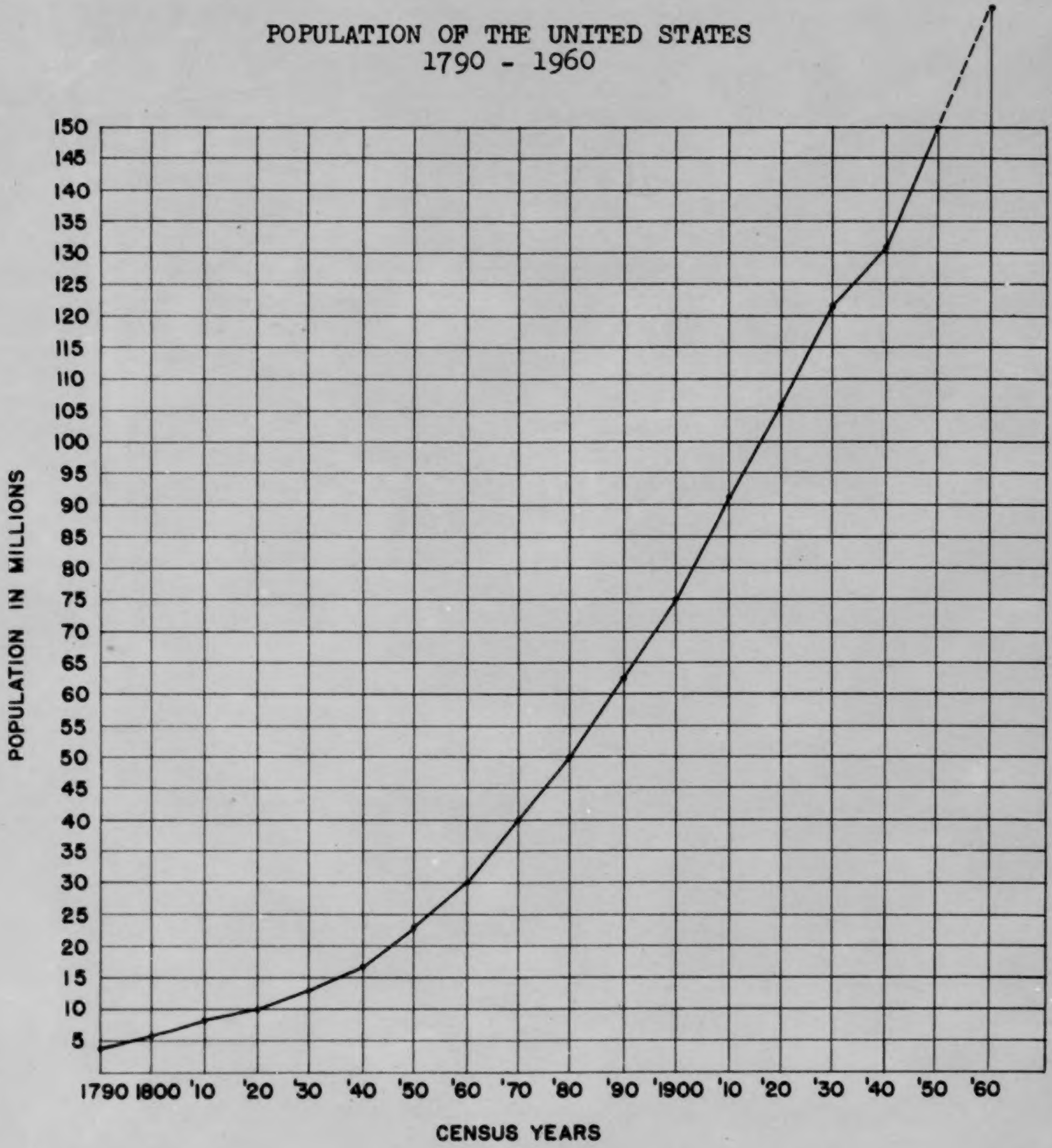


Figure 11-1

The "picture" in Figure 11-1 is called a broken line graph as many of you already know. Such graphs are usually made on cross section paper like that used in the graph in Figure 11-1. Each point in the graph represents the population for the year whose number on the base line is directly below it. For example, the point representing the population of 31.4 millions in 1860 is on the vertical line through the point labeled 1860 and on the horizontal line through the point labeled 31.4 on the vertical line at the left. Actually there is no point on the vertical line labeled 31.4 but you can tell about where it should be--about $\frac{1}{3}$ of the way from the point labeled 30 to the point labeled 35. After all the points are located, each successive pair of points is connected with a segment to get the broken line graph.

The graph shows how the population of the U. S. changed from the year 1790 to the year 1950. The predicted change for 1950-1960 is shown by the dotted line segment.

Class Discussion Questions

Use Figure 11-1 to answer the questions.

1. Did the population increase more between 1900 and 1910 than between 1800 and 1810?
2. Does the graph show a decrease in population over any ten-year period?
3. If the population for 1810 and 1820 had been the same how would this be shown in the graph?
4. What was the approximate population in 1945? in 1895? How much had it changed in the 50 years between these two dates?

5. If the population increases at the same rate from 1950 to 1960 as from 1940 to 1950 (if the graph climbs in a straight line from 1940 to 1960), what will the population be in 1960? (This is written in 1959. If the census for 1960 is known at the time you answer the question, compare the census and your answer.)

Exercises 11-2

1. Make a broken line graph to represent the data given in this table:

The number of students in Franklin Junior High School in years 1952-1957:

1952 --- 86
1953 --- 150
1954 --- 164
1955 --- 196
1956 --- 235
1957 --- 254

2. Make a broken line graph to represent the data in Table 11-3.

Table 11-3

Popular Vote in Millions Cast for Presidential Candidates of the U.S. 1928 to 1956

Year	Republican Party	Democratic Party
1928	21.4	15.0
1932	15.8	22.8
1936	16.7	27.5
1940	22.3	26.8
1944	22.0	24.8
1948	22.0	24.1
1952	33.8	27.3
1956	35.6	25.7

Observe these instructions:

(a) In the same graph draw one broken line (in blue) for the Republican party and one (in red) for the Democratic party.

(b) Find in your textbook, or elsewhere, the name of the president elected and the name of the unsuccessful candidate in each election.

(c) Use Tables 11-2 and 11-3 to find the total percent of the population who voted for either the Republican or Democratic party candidate in the presidential election in 1940.

11-3. Bar Graphs

Table 11-3 gives the pupil enrollment in the seventh grade in the U. S. for the years 1952-1959 and the expected enrollment for the years 1960-1963.

Table 11-3

Seventh Grade Pupil Enrollment in U. S. 1952-1963

Year	Enrollment in thousands
1952	2,159
1953	2,224
1954	2,354
1955	2,521
1956	2,586
1957	2,599
1958	2,707
1959	3,075
*1960	3,260
*1961	3,302
*1962	3,333
*1963	3,398

*Expected enrollment for the years 1960-1963.

The data in this table are represented by the graph in Figure 11-2. This kind of graph is called a bar graph. The years are represented along the horizontal base line and the enrollment in thousands is represented along the vertical line at the left. The bars are spaced along the base line so that the distance between any two bars is the same as the width of each bar. The number represented by each bar can be read from the vertical scale. It is the number represented by the point on the vertical scale which is in the straight line with the top of the bar. The number of the year is written at the bottom end of each bar.

SEVENTH GRADE PUPIL ENROLLMENT IN U. S.
1952 - 1963

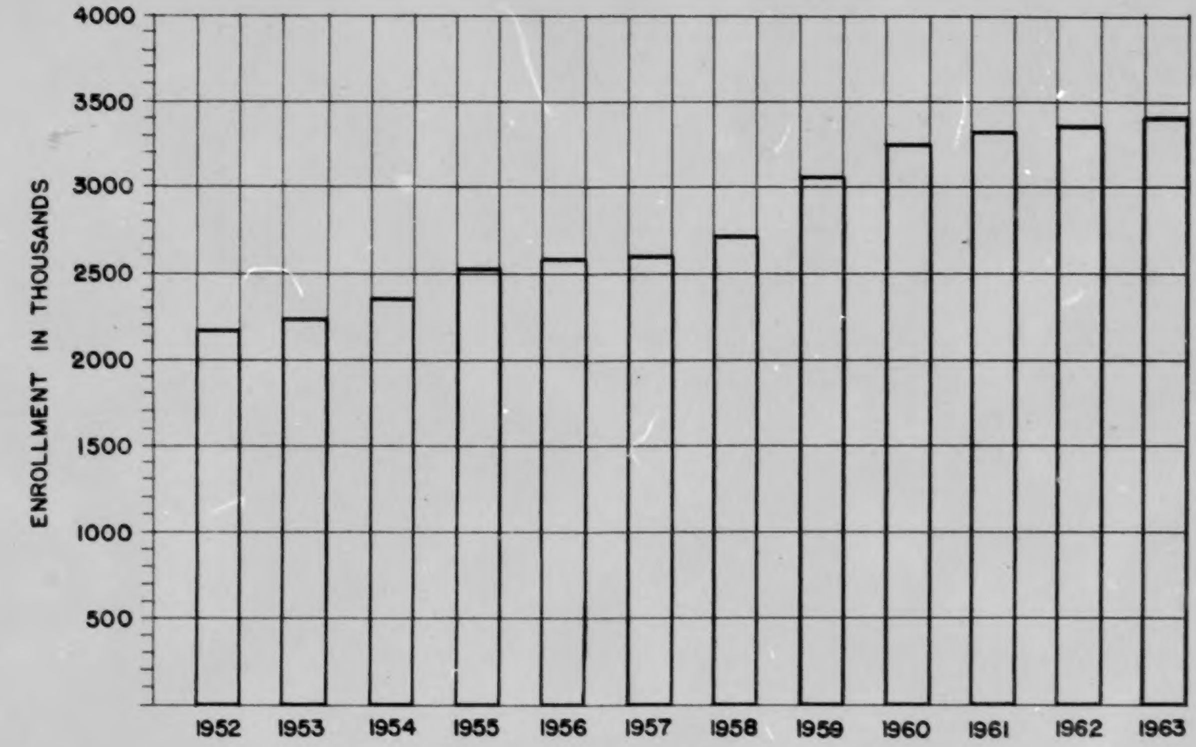


Figure 11-2

Exercises 11-3

1. If each seventh grade pupil in 1960 needs a mathematics textbook which costs $3.25 what will be the total amount spent for books to supply the whole seventh grade?

2. During which year was the enrollment about 3 million pupils?
3. Between which two years did the greatest change in enrollment occur? Use Fig. 11-2 to obtain the answer, then use Table 11-3 to see if your answer is correct.
4. Draw a bar graph to represent the number of people killed in different types of accidents during 1956 as shown in this table:

Motor vehicle accidents	40,000
Falls	20,200
Fires and injuries from fires	6,500
Drownings	6,100
Railroad accidents	2,650

11-4. Circle Graphs

A circle graph is shown in Figure 11-3. Such a graph is used to show the comparison between the parts of a whole and between the whole and any part. This graph shows the percents of income which a family spent for food, clothing, rent and miscellaneous expenses and the percent of income which was put into savings.

HOW ONE FAMILY SPENDS ITS MONEY

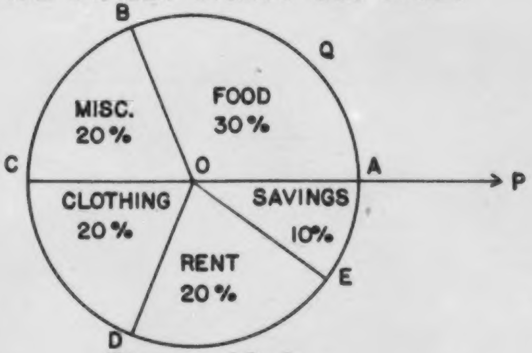


Figure 11-3

The family spent 30 percent of its income for food, 20 percent for clothing, 20 percent for rent, 20 percent for miscellaneous expenses and saved 10 percent. The family's total income is represented by the area of the circle. Any length can be chosen for the length of the radius. The length is usually chosen so that the circle is large enough to show clearly the parts into which it is divided and small enough to get it in the space.

Since the part of the income spent for food is 30 percent we must have an area which is 30 percent of the circle's area to represent this part of the income. To get this area we begin by drawing the ray OP . Label with A the intersection point of the ray and the circle. We know that we can divide the circle into 360 equal parts by drawing 360 angles of 1 degree, each with its vertex at O. Thirty percent of 360° is 108°. We draw a central angle of 108° with one side on OA. The other side is OB. The angle AOB is 30 percent of 360° and the area bounded by the closed curve OAQB is 30 percent of the area of the circle. Hence the interior of the closed curve OAQB represents the part of the income spent for food.

How many degrees are there in each of the following: $\angle$ BOC, $\angle$COD, $\angle$ DOE, $\angle$ EOA? Use a protractor to answer this question.

If the family's income is $6,000 how much is spent for food? How much is put into savings?

In a certain school there are 480 pupils. At lunch time 80 pupils go home for lunch, 120 bring their lunch, and 280 buy their lunch in the school lunchroom. The circle graph in Figure 11-4 shows the way the pupils are divided at lunch time. Before making this circle graph we had to find either what percent or what fractional part of the pupils go home for lunch, bring their lunch, and buy their lunch. These fractions and percents are shown in the table.

	Number of Pupils	Fractional Part of Total	Percent	Degrees
Go home	80	1/6	16 2/3	60
Bring lunch	120	1/4	25	90
Buy lunch	280	7/12	58 1/3	210
Total	480	1	100	360

SOURCES OF PUPIL'S LUNCHES

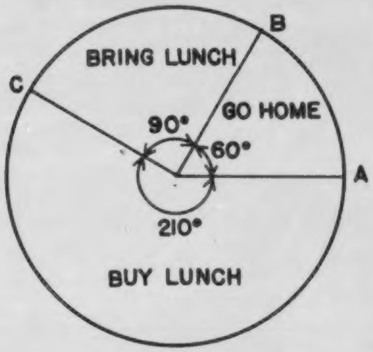


Figure 11-4

Now we find the size of $\angle$ AOB by taking 1/6 of 360° to get 60°, $\angle$ BOC by taking 1/4 of 360° to get 90°, and $\angle$ AOC by taking 7/12 of 360° to get 210°.

Exercises 11-4

Make a circle graph to represent the information in each of the following problems. Make one graph for each problem.

1. In 1949 it was found that school-related accidents which involved seventh grade pupils in the U. S. happened as shown. (Figures given are very close approximations to actual figures.)

   60 percent of the accidents happened in the school buildings
   30 percent of the accidents happened on the school playgrounds
   10 percent of the accidents happened on the way to and from school

2. In 1956 the figures for the accidents of problem 1 were as follows: (Again, figures are very close approximations.)

   36 percent of the accidents happened in the school buildings

54 percent of the accidents happened on the school playgrounds
10 percent of the accidents happened on the way to and from school.

3. There were 1955 daytime railroad-motor vehicle accidents reported in 1956. In 79% of these the locomotive hit the auto; in 17% the auto hit the locomotive, and in 4% the auto hit some other part of the train.

4. In the 1447 railroad-motor vehicles accidents reported happening at night in 1956, the locomotive hit the auto in 50% of the accidents, the auto hit the locomotive in 22%, and the auto hit some other part of the train in 28% .

11-5. Measures of Central Tendency

Some information can be determined easily by looking at all the data in table form. Sometimes, however, a large number of items in a table makes it confusing to understand. In this case it would be better to describe the set of data by using only a few numbers. Finding an average of such a set of numbers is often very helpful in studying the data given to you.

Do you know how to find an average? You have been doing this for some time, but did you know there are several kinds of averages?

Arithmetic Mean

When you calculate the average from a set of numerical grades in spelling by adding the numerical grades and dividing the sum by the number of grades, you accept a number to represent the set of grades. This useful average with which you are already familiar is called the arithmetic mean or the mean. (When the word mean is used in this chapter it always refers to the arithmetic mean.)

Let us look once more at the heights recorded in Table 11-1. This table gives a list of the heights, ordered from greatest to least, of 15 pupils.

Table 11-1

Heights of 15 Seventh Grade Pupils

Pupils	Heights in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	52

In describing this set of data, can we choose a few numbers that best represent these data? One number would be the average, called the arithmetic mean. In this table the average height (arithmetic mean) is $\frac{874 \text{ (sum of the heights)}}{15} \approx 58$. This measure of central tendency can be computed without arranging the data in any special way. It is a commonly used measure.

Median

Another way of measuring the central tendency of a set of data is finding, if possible, a number so that half of the numbers in the set are greater and half are less than the number found.

The median of a set of numbers is the middle one of the set when the numbers in the set are arranged in order of size, either from smallest to largest or from largest to smallest. In the set of heights in Table 11-1 the middle number is 59. This is the median of the set. Half of the numbers are greater than 59 and half are less. Seven pupils are taller than 59 inches and seven are shorter than 59 inches.

The median of a set of numbers is not always a number in the set. If there is an even number of numbers in the set, and the two middle ones are not equal, the median is commonly taken as the average of the two middle numbers. For example, in the set of numbers 8, 10, 11, 12, 14, 16, 17, 19 the two middle numbers are 12 and 14. The median is 13, the average of 12 and 14, although 13 is not in the set.

The set of scores 12, 13, 15, 15, 15, 15, 16, 18, 19, 20 has 10 numbers in it. The two middle ones are 15 and hence the median is 15. But the third and fourth scores are also 15 so that 15 is not a score such that 5 scores are smaller and 5 are larger than it. In this set of data it is better to say that 4 scores are 15 and two scores are smaller and four scores are greater.

In the set of salaries \$2050, \$2100, \$2300, \$2400, \$2500, \$2600, \$2700, \$2700, \$2700, \$3150 the median salary is \$2550. The arithmetic mean is \$2520. The median and arithmetic mean are nearly equal. But, if the largest salary had been \$5150 instead of \$3150 the arithmetic mean would have been \$2720 and the median would have still been \$2550. This illustrates that the usefulness of the median in describing a set of numbers often lies in the fact that one, or a few, numbers which are much larger, or much smaller, than most of the numbers in the set does not affect the median but does affect the arithmetic mean.

Mode

Which height occurs more than any other in Table 11-1? How many pupils have this height? This height is called the mode.

In sets which you have studied such as the set of natural numbers 1, 2, 3, 4, 5, ... no number occurred in the set more than once. But in a set of data some number, or numbers, may occur more than once. If a number occurs in the set of data more often than any other number it is called the mode. (We might say it is the most fashionable.) There may be several modes. In Table 11-1 there was just one. In the set of salaries \$2,050, \$2,100, \$2,300, \$2,400, \$2,500, \$2,600, \$2,700, \$2,700, \$3,150 the mode is \$2700. But in the set of scores 19, 20, 21, 21, 21, 24, 26, 26, 26, 29, 30 there are two modes 21 and 26. (These are equally fashionable.) If there had been another score of 21 in this set of scores, what would the mode have been? In Table 11-1 if the 12th pupil were 55 inches tall how would this affect the mode?

The range in a set of data is the difference between the largest and smallest numbers in the set. In the set of salaries given above the range is \$1100 and in the set of scores the range is 11.

Exercises 11-5a

1. Find the mode of the following list of chapter test scores:
   79, 94, 85, 81, 74, 85, 91, 87, 69, 85, 83.
2. From the scores in Problem 1, find the:
   (a) Mean (b) Median (c) Range

3. The following annual salaries were received by a group of ten employees:

   \$4,000, \$6,000, \$12,500, \$5,000, \$7,000, \$5,500, \$4,500, \$5,000, \$6,500, \$5,000.

   (a) Find the mean of the data

   (b) How many salaries are greater than the mean?

   (c) How many salaries are less than the mean?

   (d) Does the mean seem to be a fair way to describe the "average" of this data?

   (e) Find the median of the set of data.

   (f) Does the median seem to be a fair way to describe the "average" of this data?

4. Following are the temperatures in degrees Fahrenheit at 6 p.m. for a two week period:

   47, 68, 58, 80, 42, 43, 68, 74, 43, 46, 48, 76, 48, 50

   Find the (a) Mean (b) Median (c) Range

Grouping Data

If you were listing heights of a very large group of pupils, it might be inconvenient to list each one separately. You might group the figures in some such way as this:

Height in Inches	Number of pupils
62-64	12
59-61	17
56-58	42
53-55	57
50-52	33
47-49	14

In order to find the middle pupil, find the total number and divide by 2. The sum of 12 + 17 + 42 + 57 + 33 + 14 is 175, and $\frac{175}{2} = 87\frac{1}{2}$ so the middle person will be the 88th one, counting from the top or bottom. If we count down from the top, 12 + 17 + 42 = 71. We need 17 more to reach 88. Counting down 17 more in the group of 57 brings us to the upper part of that group. Since the 88th person is within that group, we say that the median height of the whole group of pupils is between 53 and 55 inches. Since the 88th person comes rather high in that group as we count down, we say that the median height is likely to be nearer 55 than 53.

Let's check our work and count up from the bottom to the 88th person. 14 + 33 = 47. We need 41 more than 47 to make 88, so we count 41 more and that takes us into the upper part of the group of 57 as we found when we counted down from the top. Again you find the 88th person in the group of 57 whose height is between 53 and 55. Thus the median height of the group is between 53 and 55 inches.

Exercises 11-5b

1. Give an example in which your principal might choose to group data rather than use individual numbers.
2. Find the median of the following age groups. What is the median age?

Ages in Years	Number in Group
27-29	35
24-26	48
21-23	68
18-20	18
15-17	94
12-14	53
9-11	73
6-8	26

3. Find the median by grouping the following data on temperatures: (Use intervals of 5, namely 50-54, 55-59, etc. to 90-94.) 62, 74, 73, 91, 68, 84, 75, 76, 80, 77, 68, 72, 71, 86, 82, 74, 55, 72, 50, 63, 71.

Average Deviation

Another useful method of studying a set of numbers is to find the deviation or difference of each number from the mean.

Consider the set of numbers 4, 8, 10, 4, 5, 4, 7. What is the range of this set of numbers? What is the arithmetic mean of this set?

Could you find the difference (the deviation) between the smallest number in this set and the mean? What is the deviation of the largest number in this set from the mean? The deviations of the numbers in this set from the mean are 2, 2, 4, 2, 1, 2, 1. You find the mean of these deviations to be $\frac{2 + 2 + 4 + 2 + 1 + 2 + 1}{7}$ $\frac{14}{7} = 2$. This is called the average deviation. This is still another "average" that is very helpful in studying data. Let us see how this measure, the average deviation, helps to throw light on a set of data.

The total receipts of the federal government in the years 1946-1955 were as follows:

Year	Billions	Deviations from Mean
1946	44	-11.5
1947	45	-10.5
1948	46	- 9.5
1949	43	-12.5
1950	41	-14.5
1951	53	- 2.5
1952	68	+12.5
1953	73	+17.5
1954	73	+17.5
1955	69	+13.5
Total	555	

The arithmetic mean of these receipts is the total, 555, divided by 10, or 55.5.

The third column shows the deviation of each year's receipts from the mean, 55.5. The -11.5 means that 44 is 11.5 below the mean, for instance, and +13.5 means that 69 is 13.5 above the mean. Have you seen minus signs used for temperature in a way similar to those in this table? Give other examples of minus signs used similarly.

Exercises 11-5c

(The first three questions in these Exercises refer to the data for the receipts of the federal government in 1946-1955.)

1. In which year was the deviation from the mean the greatest?
2. In which year was the deviation from the mean the least?
3. Find the average deviation by finding the total of the deviations and dividing by 10. The signs before the numbers are disregarded since we want to know the size of the deviations, no matter which direction they are from the mean.
4. Find, to the nearest tenth, the mean and the average deviation of the following test scores:

   85, 82, 88, 76, 90, 84, 80, 82, 84, 83.

5. Find the mean and the average deviation of the test scores (same test, but in another class):
94, 84, 68, 74, 98, 70, 96, 84, 76, 96.

*6. Another method of calculating the arithmetic mean is shown in the following example.

Example. Calculate the arithmetic mean of the set of scores:

10, 11, 13, 15, 19, 20, 21, 21, 23

We begin by making a reasonable guess of some number for the mean. Suppose we guess 18. Next we find the deviation of each score in the set from 18.

Scores	Deviations from 18
10	-8
11	-7
13	-5
15	-3
19	+1
20	+2
21	+3
21	+3
23	+5

The sum of the deviations of the scores less than 18 is 23. The sum of the deviations of the scores greater than 18 is 14. We take the difference 23 - 14 = 9 and divide this by the number of scores which is 9: 9 ÷ 9 = 1. Since the deviations from 18 were greater for the scores less than 18 than for the scores greater than 18 we subtract 1 from 18 to get the correct mean of 17.

Use the method of the above example to find the mean of the set of scores: 40, 43, 44, 47, 48, 49, 51.

The numbers which we have been finding for the range, mean, median, mode, and average deviation from the mean for a set of data are called the measures of central tendency. This name is given to them since by knowing them we can tell whether the numbers in the data bunch together or scatter out. All of these measures of central tendency are illustrated by use of the following set of salaries of 12 people and their representation in the broken line graph in Figure 11-5. Salaries: \$4,000, \$4,500, \$4,500, \$5,000, \$5,000, \$5,000, \$5,250, \$5,250 , \$5,250, \$5,250, \$5,500, \$6,000.

SALARIES OF A SELECTED GROUP

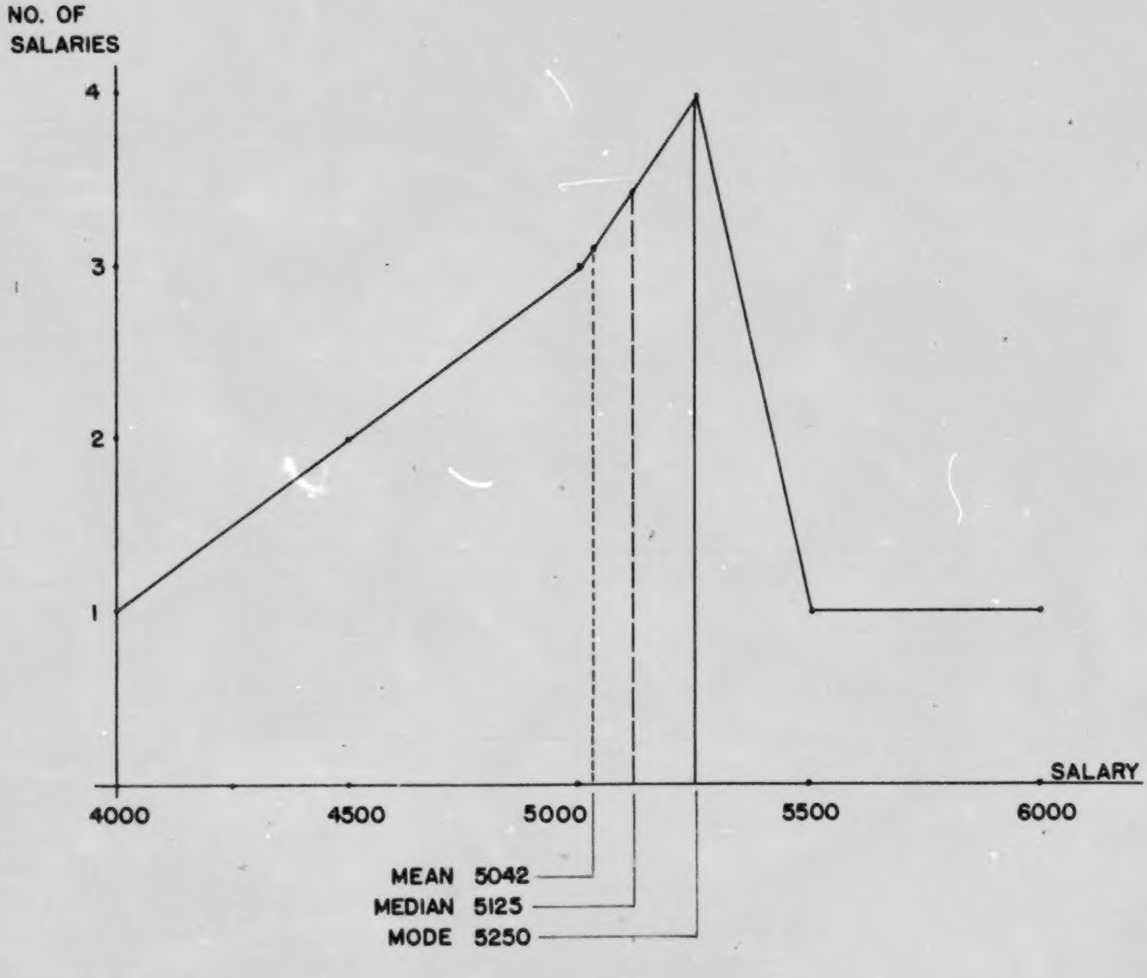


Figure 11-5

Calculation of average deviation from the mean, \$5,042 (to the nearest dollar).

Salaries	Deviation from \$5,042
\$4,000	-\$1042
4,500	- 542
4,500	- 542
5,000	- 42
5,000	- 42
5,000	- 42
5,250	+ 208
5,250	+ 208
5,250	+ 208
5,250	+ 208
5,500	+ 458
6,000	+ 958
Total	\$4500

$\frac{\$4500}{12}$ = \$375, average deviation from the mean.

Range: \$6000 - \$4000 = \$2000.

The locations of the lines which represent the mean, median, and mode show that these numbers are nearly equal and the graph shows the salaries are about equally distributed on both sides of these lines.

11-6. Sampling

We all know that a presidential election is held every four years in the United States. In which year will the next one be held? People are very much interested in the outcome of the elections. Sometimes, long before the elections are held, organizations make predictions concerning who will be elected. These organizations not only predict who will be elected but even predict the percent of the votes cast that each candidate will receive. The candidates and the percent of vote predicted for each of them in the election of 1948 by three different polls is shown in the table.

Candidates	Dewey	Truman	Thurmond	Wallace
Poll No. 1	49.5%	44.5 %	2 %	4 %
Poll No. 2	49.9%	44.8 %	1.6 %	3.3 %
Poll No. 3	52.2%	37.1 %	5.2 %	4.3 %

In the election the percent of vote for each was: Truman 49.5%, Dewey 45.1%, Thurmond 2.4%, Wallace 2.4%. Do you see why this election is called the "surprise election"?

Although neither poll predicted the election correctly, their predictions were close. How did they do it? Did they go about the U. S. and ask every voter how he was going to vote? Or, did they write each voter a letter? Either of these ways would have been very expensive and would have taken a great deal of time. Instead of either of these they used a method called sampling. This means that the organizations who made the predictions selected a "sample" of the population of the U. S. Then, after asking the people in the "sample" how they would vote, the organizations predicted that the vote in the entire country would be in the same ratio as the vote in the "sample".

If you have ever had a blood-count, the doctor took a very small amount of blood from the tip of your finger, or from your ear lobe, and then counted the red and white corpuscles in it. This was a very small sample of your blood. The count in it was taken as a reliable representation of the count in all the blood in your body. Perhaps you can think of other examples of sampling.

Let us suppose you know that all the employees whose names are listed in the employee's directory of a certain large manufacturing firm are men over twenty-one years of age. Then let us ask how we might use the sampling method to make an estimate of the average height of these men. You might select the first and last man listed in the directory and find their average height. Or, you might select the first name listed under each letter of the alphabet, or the last name under each letter, or both the first and last names under each letter. There are many ways a sample could be selected. Some would be good and some would be bad. Do you see any objection to any of the methods suggested? The way of selecting a sample so that it will be a good representation of the group from whom the sample is selected is a very difficult part of the job.

As a project for your class you might find the average height of the boys, or girls, in the class by use of the sampling method. Choose two or three different samples. One sample might be obtained by selecting the boys whose first names have the same letter. Another sample might be obtained by choosing the boys whose birthdays occur in certain months, for example, in March, July, November. Then after measuring the height of the boys in each sample find the average height. After all of this is done you might find the average height of all the boys in the class and compare this with the average found in each sample. The comparisons would show which sample was the best representation of the entire group of boys in the class for determining the average height.

11-7. Summary

The subject matter of statistics deals, in part, with collecting data, putting it in table form, and representing it by graphs. The tabulating and graphing of the data should be done in such ways that the story told by the data can be interpreted and summarized easily. The broken line graphs, bar graphs, and circle graphs are just a few of the kinds of graphs that may be used.

You have learned that there are many different measures for the central tendency of the same set of data. The next time you see graphs or tables of a set of figures in the magazines, newspapers, or your social studies book, look them over carefully. If averages are mentioned, see if you can tell which average is used. Whenever the kind of average used is not stated, you have a right to question whether the average used gives the best representation of all the data.

To help you recall the new terms you have used in working with statistics, they are listed for you:

Range -- difference between largest and smallest number in a set.

Arithmetic mean or mean -- the sum of all the numbers in a set divided by the number in the set.

Median -- the middle number when data are ordered either from smallest to largest or largest to smallest. When there is no one middle number, the average of the two middle numbers is the median.

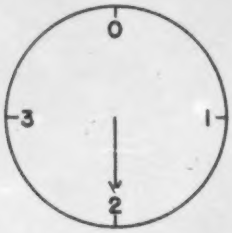
Mode -- the number occurring most in the list of data. There may be several modes.

Average deviation -- average of the deviations from the mean.

CHAPTER 12

MATHEMATICAL SYSTEMS

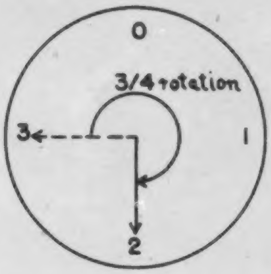
12-1. A New Addition



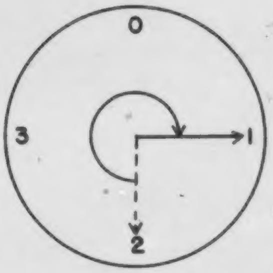
The sketch above represents the face of a four-minute clock. Zero is the starting point and, also, the end-point of a rotation of the hand.

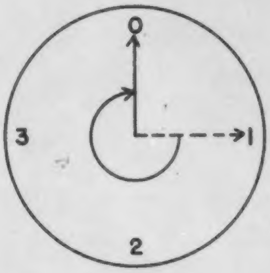
With the model we might start at 0 and move to a certain position (numeral) and then move on to another position just like the moving hand of a clock. For example, we may start with 0 and move $\frac{2}{4}$ of the distance around the face. We would stop at 2. If we follow this by a $\frac{1}{4}$ rotation (moving like the hand of a clock), we would stop at 3. After a rotation of $\frac{2}{4}$ from 0 we could follow with a $\frac{3}{4}$ rotation. This would bring us to 1. The first example could be written 2 + 1 gives 3 where the 2 is $\frac{2}{4}$ of a rotation from 0, the + means to follow this by another rotation (like the hand of a clock), and the 1 means $\frac{1}{4}$ rotation, thus we arrived at the position marked 3 (or $\frac{3}{4}$ of a rotation from 0). The second example would be 2 + 3 gives 1 where the 2 and + still mean the same as in the first example and the 3 means a rotation of $\frac{3}{4}$. A common way to write this is: $2 + 3 \equiv 1 \pmod 4$. This is read "Two plus three is equivalent to one, mod 4." Mod 4 tells how many numerals there are on the face of the clock (that is, 0, 1, 2, and 3) and the system is called the mod 4 system. Though this notation may be meaningless to you now in Section 12-8 this will be explained more thoroughly.

Example 1. Find 3 + 3 (mod 4).



Example 2. Find (2 + 3) + 3 (mod 4).





$$2 + 3 \equiv 1 \pmod{4} \qquad 1 + 3 \equiv 0 \pmod{4}$$

$$(2 + 3) + 3 \equiv 1 + 3 \equiv 0 \pmod{4}$$

The following table illustrates some of the addition facts in the mod 4 system.

Mod 4

+	0	1	2	3
0	0	1	2	
1			3	0
2				1
3		0		

We read a table of this sort by following across horizontally from any entry in the left column, for instance 2, to the position below some entry in the top row, such as 3. The entry in this position in the table is then taken as the result of combining the element in the left column with the element in the top row (in that order). In the case above we write $2 + 3 \equiv 1 \pmod 4$. Use the table to check that $3 + 1 \equiv 0 \pmod 4$.

Example 3. Complete the following number sentences to make them true statements.

(a) $3 + 4 \equiv ? \pmod 5$

The mod 5 system represented by the face of a clock should have five positions; namely, 0, 1, 2, 3, and 4.

$3 + 4 \equiv 2 \pmod 5$

The 3 means a rotation of $\frac{3}{5}$ from 0. This is followed by a $\frac{4}{5}$ rotation which ends at 2.

(b) $2 + 3 \equiv ? \pmod 5$

$2 + 3 \equiv 0 \pmod 5$. This is a $\frac{2}{5}$ rotation from 0 followed by a $\frac{3}{5}$ rotation which brings us to 0.

(c) $4 + 3 \equiv ? \pmod 6$

In the mod 6 system, the positions on the face of the clock are marked 0, 1, 2, 3, 4, and 5.

$4 + 3 \equiv 1 \pmod 6$

If you cannot readily understand this, a sketch of a clock will help you.

Exercises 12-1

1. Copy and complete the table for addition mod 4.
2. Complete the following number sentences to make them true statements.

(a) $3 + 3 \equiv ? \pmod 5$ (c) $5 + 3 \equiv ? \pmod 6$

(b) $4 + 3 \equiv ? \pmod 5$ (d) $5 + 5 \equiv ? \pmod 6$

(e) $3 + 6 \equiv ?\ (\text{mod } 7)$

*(f) $3 + 3 \equiv 1\ (\text{mod } ?)$

*(g) $4 + 4 \equiv 2\ (\text{mod } ?)$

*(h) $3 + 3 + 2 \equiv 1\ (\text{mod } ?)$

3. Make a table for addition mod 3 and for addition mod 5.

4. Find a replacement for x to make each of the following number sentences a true statement.

(a) $4 + x \equiv 0\ (\text{mod } 5)$

(b) $x + 1 \equiv 2\ (\text{mod } 3)$

(c) $1 + x \equiv 2\ (\text{mod } 3)$

(d) $2 + x \equiv 4\ (\text{mod } 5)$

(e) $3 + x \equiv 2\ (\text{mod } 5)$

(f) $x + 4 \equiv 3\ (\text{mod } 5)$

(g) $x + 2 \equiv 0\ (\text{mod } 3)$

(h) $4 + x \equiv 4\ (\text{mod } 5)$

5. You have a five-minute clock. How many complete revolutions would the hand make if you were using it to tell when 23 minutes had passed? Where would the hand be at the end of the 23 minute interval? (Assume that the hand started from the 0 position.)

12-2. What is an Operation?

We are familiar with the operations of ordinary arithmetic--addition, multiplication, subtraction and division of numbers. In the preceding section, a different operation was discussed. We made a table for the new addition of the numbers 0, 1, 2, 3. This operation is completely described by the table that you made in Problem 1 of Exercises 12-1. The table tells what numbers can be put together by the new addition, and it also tells what is the result when two numbers are put together. For instance, the table tells us that the number 5 cannot be put together with any number in the new addition since "5" does not appear in the left column nor in the top row. It also tells us that $2 + 3 \equiv 1\ (\text{mod } 4)$.

Study the following tables:

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

From each one of these tables we can find a certain set (the set of elements in the left column and top row) and we can put any two elements of this set together to get one and only one thing. For instance, in Table (a), the set is {1, 2, 3, 4, 5} since these are the numbers which appear in the left column and top row. These are the only numbers which can be put together by Table (a). In Table (b), the set is {3, 5, 7, 9}. What set is given by Table (c)? by Table (d)?

Here are some examples from the tables:

3 + 5 = 3 in Table (a)

3 + 5 = 8 in Table (b)

2 □ 1 = 5 and 2 □ 2 = 6 in Table (c)

1 ⊙ 1 = 3 in Table (d).

Definition. A binary operation defined on a set is a way of combining any two elements of the set to get a definite thing.

The two elements we combine may be the same one, and the result of the operation (the "definite thing" which we get) may or may not be an element of the set.

You are already familiar with some operations defined on the set of whole numbers.

Any two whole numbers can be added. Addition of 8 and 2 gives 10.

Any two whole numbers can be multiplied. Multiplication of 8 and 2 gives 16.

Addition and multiplication are two different operations defined on the set of whole numbers.

In discussing subtraction, for instance with whole numbers, it is convenient to look ahead to the work of the eighth grade. The expression "6 - 9" is not the name of anything you have studied in this course. That is, it is not now possible for us to combine 6 and 9 (in that order) by subtraction and get "a definite thing" and you may wonder whether or not subtraction is an operation. Next year you will learn that there is "a definite thing" (in fact, a number) which is called "6 - 9". Because of this, we will consider that subtraction is a binary operation defined on the whole numbers (or rational numbers, etc.), even though we are not yet acquainted with all the results obtained from subtraction.

When an operation is described by a table, the elements of the set are written in the same order in the top row (left to right) and in the left column (top to bottom). Keeping the order the same will make some of our later work easier.

We must also be careful about the order in which two elements are combined. For example,

$2 \square 1 = 5$, but $1 \square 2 = 4$.

For this reason, we must remember that when the procedure for reading a table was explained, it was decided to write the element in the left column first and the element in the top row second with the symbol for the operation between them. We must examine each new operation to see if it is commutative and associative. These properties have been discussed in previous chapters; they are briefly reviewed here.

An operation $+$ defined on a set is called commutative if any elements, a, b, of the set can be put together as $a + b = b + a$.

An operation $+$ defined on a set is called associative if any elements, a, b, c, of the set can be combined as $(a + b) + c$, and also as $a + (b + c)$, and the two results are the same: $(a + b) + c = a + (b + c)$.

Exercises 12-2

1. Use the tables in this section to answer the following questions:
   - (a) $3 + 3 = ?$ if we use Table (a).
   - (b) $3 + 3 = ?$ if we use Table (b).
   - (c) $3 \square 2 = ?$
   - (d) $2 \square 3 = ?$
   - (e) $2 \odot 2 = ?$
   - (f) $1 \odot 1 = ?$
   - (g) $(2 \odot 3) \odot 3 = ?$
   - (h) $2 \odot (3 \odot 3) = ?$
   - (i) $(1 \square 1) \square 2 = ?$
   - (j) $1 \square (1 \square 2) = ?$

2. Which of the binary operations described in the tables in this section are commutative? Is there an easy way to tell if an operation is commutative when you examine the table for the operation? What is it?

3. Can you tell if an operation is associative when you examine a table for the operation? Do you think the operations described in the tables in this section are associative?

4. Are the following binary operations commutative? Make at least a partial table for each operation. Which ones do you think are associative?

   (a) Set: All counting numbers between 25 and 75.
   Operation: Choose the smaller number.
   Example: 28 combined with 36 produces 28.

   (b) Set: All counting numbers between 500 and 536.
   Operation: Choose the larger number.
   Example: 520 combined with 509 produces 520.

   (c) Set: The prime numbers.
   Operation: Choose the larger number.

   (d) Set: All even numbers between 39 and 61.
   Operation: Choose the first number.
   Example: 52 combined with 46 produces 52.

   (e) Set: All counting numbers less than 50.
   Operation: Multiply the first by 2 and then add the second.
   Example: 3 combined with 5 produces 11 ($2 \cdot 3 + 5 = 11$).

   (f) Set: All counting numbers.
   Operation: Find the greatest common factor.
   Example: 12 combined with 18 produces 6.

(g) Set: All counting numbers.

Operation: Find the least common multiple.

(h) Set: All counting numbers.

Operation: Raise the first number to a power whose exponent is the second number.

Example: 5 combined with 3 produces 5^3.

5. Make up a table for an operation that has the commutative property.

6. Make up a table for an operation that does not have the commutative property.

We have been discussing binary operations. The word "binary" indicates that two elements are combined to produce a result. There are other kinds of operations. A result might be produced from a single element, or by combining three or more elements. When we have a set and, from any one element of the set, we can determine a definite thing, we say there is a unary operation defined on the set.

*7. Can you think of a way of describing the following unary operation by some kind of a table?

Set: All the whole numbers from 0 to 10.

Unary Operation: Cube the number.

Example: Doing the operation to 5 produces $5^3 = 125$.

12-3. Closure

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

Study the Tables (a) and (b). One interesting difference between the tables is this: In Table (a), the results of performing the operation (that is, the numbers written in the table) are always elements of the set on which the operation is defined (the set of elements which appear in the left column and in the top row). In Table (b), this is not true. For instance, $5 + 7 = 12$ in Table (b), and 12 is not an element of the set $\{3, 5, 7, 9\}$ on which the operation of Table (b) is defined.

We recall that a set is closed under an operation if it is always possible to combine any two elements of the set and if the result obtained is always an element of that set. The two elements we start with may be the same one.

Example 1. The set of even numbers is closed under addition. This means that if any two even numbers are added, the result is always an even number. $2 + 2 = 4$ (We used the same number.) $14 + 6 = 20$ $44 + 86 = 130$.

Example 2. The set of odd numbers is not closed under addition. This means that if two odd numbers are added, the result is not always an odd number. For example, $3 + 5 = 8$. Is one example enough to show that the set of odd numbers is not closed under addition? We could give more examples; actually, the sum of two odd numbers is never an odd number.

Example 3. The set of whole numbers is not closed under subtraction. For example, consider the two whole numbers 6 and 9. There are two different ways we can put these two numbers together using subtraction: $9 - 6$ and $6 - 9$. The first numeral, "9 - 6", is a name for the whole number 3, but the numeral "6 - 9" is not the name of any whole number. Thus, subtracting two whole numbers does not always give a whole number.

Example 4. The set of counting numbers is not closed under division. It is true that $\frac{8}{2}$ is a counting number, but there is no counting number $\frac{9}{2}$. Can you give some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Exercises 12-3

1. Study again Tables (a) - (d) in this section. Which tables determine a set that is closed under the operation? Which tables determine a set that is not closed under the operation? How do you know?

2. Which of the sets below are closed under the corresponding operations?
   (a) The set of even numbers under addition.
   (b) The set of even numbers under multiplication.
   (c) The set of odd numbers under multiplication.
   (d) The set of odd numbers under addition.
   (e) The set of multiples of 5 under addition.
   (f) The set of multiples of 5 under subtraction.
   *(g) The set {0, 1, 2, 3} under subtraction mod 4. Remember that subtraction is the inverse operation of addition.
   (h) The set of counting numbers less than 50 under the operation of choosing the smaller number.
   (i) The set of prime numbers under addition.
   *(j) The set of numbers whose numerals in base five end in "3", under addition.

*3. If an operation defined on a set is commutative, must the set be closed under the operation?

*4. If an operation defined on a set is associative, must the set be closed under the operation?

*5. Make up a table for an operation defined on the set {0, 43, 100} so that the set is closed under the operation.

*6. Make up a table for an operation defined on the set {0, 43, 100} so that the set is not closed under the operation.

12-4. Identity Element; Inverse of an Element

In our study of the number one in ordinary arithmetic, we observed that the product of any number and 1 (in either order) gave that same number; that is, the product of any number and 1 is the number. For instance

$2 \times 1 = 2, \quad 1 \times 2 = 2, \quad 156 \times 1 = 156, \quad 1 \times 156 = 156.$

For any number n in ordinary arithmetic, $n \cdot 1 = n$ and $1 \cdot n = n$.

In our study of the number zero in ordinary arithmetic we observed that the sum of 0 and any number (in either order) gave that same number; that is the sum of any number and 0 is the number. For instance

$2 + 0 = 2, \quad 0 + 2 = 2, \quad 468 + 0 = 468, \quad 0 + 468 = 468$

For any number n in ordinary arithmetic, $n + 0 = n$ and $0 + n = n$.

One is the identity for multiplication in ordinary arithmetic.

Zero is the identity for addition in ordinary arithmetic.

Definition. An identity (or identity element) for a binary operation, *, defined on a set is an element, e, in the set, such that $e * a = a$ and $a * e = a$ for every element a of the set.

What is the identity for the arithmetic of the 4-minute clock?

If we add two things and get the identity for addition, then we call them additive inverses of each other. For example, in the table

0 is the identity

$2 + 2 \equiv 0 \pmod{4}$

$3 + 1 \equiv 0 \pmod{4}$

$1 + 3 \equiv 0 \pmod{4}$

Mod 4

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

These pairs of numbers, 2 and 2; 3 and 1; are said to be inverses of each other. Each element of this set has an inverse. The inverse of 0 is 0; the inverse of 1 is 3; the inverse of 3 is 1.

Definition. Two elements a and b are inverses (or either one is the inverse of the other) under a binary operation * with identity element e if a * b = e and b * a = e.

Exercises 12-4

1. Study tables (a) - (d) in Section 12-3.
   (a) Which tables describe operations having an identity and what is the identity?
   (b) Pick out pairs of elements which are inverses of each other under these operations. Does each member of the set have an inverse?
2. For each of the operations of Problem 4, Exercises 12-2;
   (a) Does the operation have an identity and, if so, what is it?
   (b) Pick out pairs of elements which are inverses of each other under these operations.
   (c) For which operations does each element have an inverse?

*3. Can there be more than one identity element for a given binary operation?

12-5. What is a Mathematical System?

The idea of a set has been a very convenient one in this book--some use has been made of it in almost every chapter. But there is really not a great deal that can be done with just a set of elements. It is much more interesting if something can be done with the elements (for instance, if the elements are numbers, they can be added or multiplied). If we have a set and an operation defined on the set, it is interesting to find out how the operation behaves. Is it commutative? associative? Is there an identity element? Does each element have an inverse? The "behavior" of the arithmetic operations (addition, subtraction, multiplication and division) on numbers was discussed in Chapters 3 and 6. We have seen that different operations may "behave alike" in some ways (both commutative, for instance). This suggests that we study sets with operations defined on them to see what different possibilities there are. It is too hard for us to list all the possibilities, but some examples will be given in this section and the next. These are examples of mathematical systems.

Definition. A mathematical system is a set of elements together with one or more operations defined on the set.

The elements do not have to be numbers. They may be any objects whatever. Some of the examples below are concerned with letters or geometric figures instead of numbers.

Example 1. Let's look at egg-timer arithmetic--arithmetic mod 3.

(a) There is a set of elements the set of numbers {0, 1, 2}

(b) There is an operation + mod 3, defined on the set {0, 1, 2}.

Mod 3

+	1	2	0
1	2	0	1
2	0	1	2
0	1	2	0

Therefore, egg-timer arithmetic is a mathematical system. Does this system have any interesting properties?

(c) The operation, + mod 3, has the commutative property. Can you tell by the table? If so, how? We can check some special cases too. 1 + 2 $\equiv$ 0 (mod 3) and 2 + 1 $\equiv$ 0 (mod 3), so 1 + 2 $\equiv$ 2 + 1 (mod 3).

(d) There is an identity for the operation + mod 3 (the number 0).

(e) Each element of the set has an inverse for the operation + mod 3.

Study the following tables.

(a)

O	A	B
A	A	B
B	A	B

(b)

*	P	Q	R	S
P	R	S	P	Q
Q	S	R	Q	P
R	P	Q	R	S
S	Q	P	S	R

(c)

~	△	□	○	╲
△	△	□	○	╲
□	□	○	╲	△
○	○	╲	△	□
╲	╲	△	□	○

Exercises 12-5

1. Which one, or ones, of the Tables (a), (b), (c) describes a mathematical system? Show that your answer is correct.
2. Use the tables above to complete the following statements correctly.

(a) B o A = ?

(b) $\triangle \sim \bigcirc = ?$

(c) $\backslash \sim \backslash = ?$

(d) A o B = ?

(e) Q * R = ?

(f) R * S = ?

(g) P * R = ?

(h) $\square \sim \bigcirc = ?$

(i) $\backslash \sim \square = ?$

(j) B o B = ?

(k) A o A = ?

(l) S * S = ?

3. Which one, or ones, of the binary operations o, *, ~ is commutative? Show that your answer is correct.

4. Which one, or ones, of the binary operations o, *, ~ has an identity element? What is it in each case?

5. Use the tables above to complete the following statements correctly.

(a) P * (Q * R) = ?

(b) (P * Q) * R = ?

(c) P * (Q * S) = ?

(d) (P * Q) * S = ?

(e) (R * P) * S = ?

(f) R * (P * S) = ?

(g) $\triangle \sim (\triangle \sim \backslash) = ?$

(h) $(\triangle \sim \triangle) \sim \backslash = ?$

(i) $(\bigcirc \sim \square) \sim \triangle = ?$

(j) $\bigcirc \sim (\square \sim \triangle) = ?$

6. Does either of the operations described by Table (b) or Table (c) seem to be associative? Why? How could you prove your statement? What would another person have to do to prove you wrong?

7. BRAINBUSTER. For each of the following tables, tell why it does not describe a mathematical system.

(a)

*	1	2
1	1	1
2	1	

(b)

*	1	2
1	the product of 3 and 6	the sum of 2 and 4
2	a number between 3 and 8	0

(c)

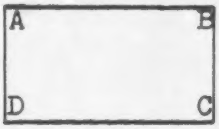
*	1	2
3	1	4
4	2	3

12-6. Mathematical Systems without Numbers

In the last section there were some examples of mathematical systems without numbers in them. Suppose we invent one. What do we need?

We must have a set of things. Then, we need some kind of an operation--something that can be done with any two elements of our set. We have found that the properties of closure, commutativity, associativity, etc. are very helpful in simplifying expressions. It would be nice to have some of these properties.

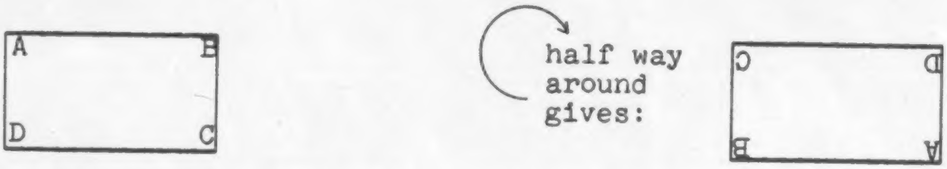
Let's start with a card. Any rectangular shaped card will do. We will use it to represent the interior of a rectangle. Lay the card on your desk and label the corners as in the sketch. Now pick the card up and write the letter "A" on the other side (the side that was touching the desk) behind the "A" you have already written. Be sure the two letters "A" are back-to-back so they are labels for the same corner of the card. Similarly, label the corners B, C, and D on the other side of the card (be sure they're back-to-back with the B, C, and D you have already written).



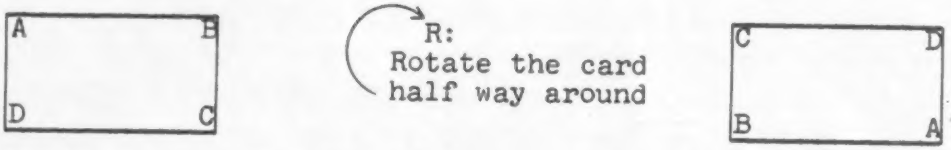
What set shall we take? Instead of numbers, let us take elements which have something to do with the card. Start with the card in the center of your desk and with the long sides of the card parallel to the front of your desk. Now move the card--pick it up, turn it over or around in any way--and put it back in the center of your desk with the long sides parallel to the front of your desk. The card looks just the same as it did before, but the corners may be labelled differently (a corner that started at the top may now be at the bottom, for instance). The position of the card has been changed, but the interior of the rectangle looks the way it did in the beginning. (The "picture" stays the same. Individual points may be moved.) The elements of our set will be these changes of position. We will take all the changes of position that make the interior of our rectangle look the way it did in the beginning. (Long sides parallel to the front of the desk.) How many of these changes are there?

We may start with the card in some position which we will call the standard position. Suppose it looks like the figure above.

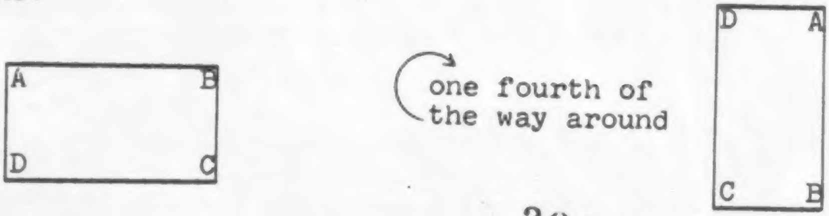
Leaving the card on your desk, rotate it half way around its center. A diagram of this change is:



Since the letters "A", "B", etc. are only used as a convenience to label the different corners of the card, we will not bother to write them upside down. The diagram below represents this change of position, and we will call the change "R" (for rotation).

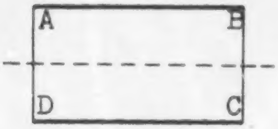


What would happen if the card rotated one fourth of the way around?

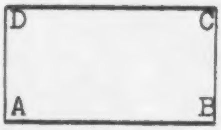


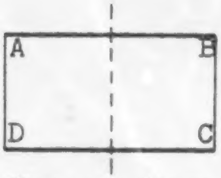
Does the card look the same before and after the change? No, this change of position cannot be in our set, since the two pictures are quite different.

Are there other changes of position of the interior of the rectangle which make it look the way it did in the beginning? Yes, we can flip the card over in two different ways as shown by the diagrams below:

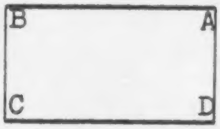


H:
Flip the card over, using a horizontal axis.





V:
Flip the card over, using a vertical axis.

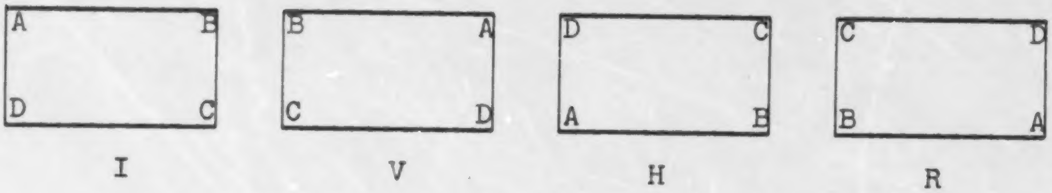


Now you know why you had to label both sides of the card so carefully. Remember, the card only represents a geometric figure for us. Turning over a card makes it different--you see the other side; but turning over the interior of a rectangle would not make it different (of course, some of the individual points would be in different positions, but the whole geometric figure would look just the same).

There is one more change of position which we must consider. It is the change which leaves the card alone (or puts each individual point back in place). Let us call it "I".

I:
Leave the card in place

Since, in our diagrams above, we have always started with the rectangle in the standard position, each change of position can be represented by the final position. The four changes are diagrammed below for easy reference.



Now we have our set; it is {I, V, H, R}. We can start a table for the operation in our mathematical system. But the table cannot be filled in yet--we have no operation! What operation shall we use? How can we "combine any two elements of our set" to get a "definite thing"? Could the names of any objects at all be put in the operation table if we want our set to be closed under the operation?

	I	V	H	R
I				
V				
H				
R				

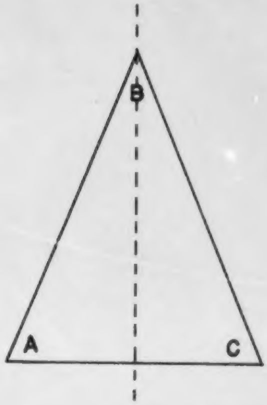
Here is a way of combining any two elements of our set. We will do one of the changes AND THEN do the other one. We will use the symbol "ANTH" for this operation (perhaps you can think of a better one). Thus "H ANTH V" means "flip the card over, using a horizontal axis, and then flip the card over, using a vertical axis." Start with the card in the standard position and do these changes to it. What is the final position of the card? Is the result of these two changes the same as the change R? What does "V ANTH H" mean? Try it with your card. Now we can fill in the table for our operation. Some of the entries are given in the table at the right.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Exercises 12-6

1. Check the entries that are given in the table above and find the others. Use your card.
2. From your table for the operation ANTH , or by actually moving a card, fill in each of the blanks to make the equations correct.
   (a) R ANTH H = ?
   (b) R ANTH ? = H
   (c) ? ANTH R = H
   (d) ? ANTH H = R
   (e) (R ANTH H) ANTH V = ?
   (f) R ANTH (H ANTH V) = ?
   (g) (R ANTH H) ANTH ? = V
   (h) (R ANTH ?) ANTH V = H
   (i) (? ANTH H) ANTH V = R
3. Examine the table for the operation ANTH.
   (a) Is the set closed under the operation?
   (b) Is the operation commutative?
   (c) Do you think the operation is associative? Use the operation table to check several examples.
   (d) Is there an identity for the operation ANTH?
   (e) Does each element of the set have an inverse under the operation ANTH?

4. Here is another system of changes. Take a triangle with two equal sides. Label the corners as in the sketch (both sides, back-to-back). The set for the system will consist of two changes. The first change, called I, will be: Leave the triangle in place. The second change, called F, will be: Flip the triangle over, using the vertical axis. F ANTH I will mean: Flip the triangle over, using the vertical axis, and then leave the triangle in place. How will the triangle look--as if it had been left in place, I, or as if the change F had been done? What does I ANTH F mean? Does F ANTH I = F or does F ANTH I = I?



(a) Complete the table below:

ANTH	I	F
I		
F		

(b) Is the set closed under this operation?

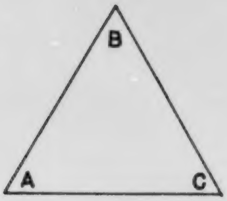
(c) Is the operation commutative?

(d) Is the operation associative? Are you sure?

(e) Is there an identity for the operation?

(f) Does each element of the set have an inverse under the operation?

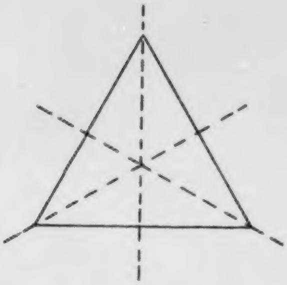
5. Make a triangular card with three equal sides and label the corners as in the sketch (both sides, back-to-back). The set for this system will be made up of these six changes.



I: Leave the triangle in place.

R: Rotate the triangle clockwise $\frac{1}{3}$ of the way around.

S: Rotate the triangle clockwise $\frac{2}{3}$ of the way around.



T: Flip the triangle over, using a vertical axis.

U: Flip the triangle over, using an axis through the lower right vertex.

V: Flip the triangle over, using an axis through the lower left vertex. Three of these will be rotations about the center (leave in place and two others). The other three will be flips about the axes. Caution: The axes are stationary; they do not rotate with the triangle. For example, the vertical axis remains vertical--it would go through a different corner of the card after rotating the triangle one third of the way around its center. Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

*6. Try making a table of changes for a square. There are eight changes. What are they? Is there an identity change? Is the operation commutative?

12-7. The Counting Numbers and the Whole Numbers

The mathematical systems that we have studied so far in this chapter are composed of a set and one operation. Examples are modular addition and the changes of a rectangle or triangle. A mathematical system given by a set and two operations would appear to be more complicated than these examples. However, as you may have guessed, ordinary arithmetic is also a mathematical system and we know that we can do more than one operation using the same set of numbers--for example, we can add and multiply. Can you name three different sets of numbers considered in ordinary arithmetic? Can you name more than three?

To be definite, let us choose the set of rational numbers. This set, together with the two operations of addition and multiplication forms a mathematical system which was discussed in chapters 6 and 8. Are there properties of this system which are entirely different from those we have considered in systems with only one operation? Yes, you are familiar with the fact that $2 \cdot (3 + 5) = (2 \cdot 3) + (2 \cdot 5)$. This is an illustration of the distributive property. More precisely, it illustrates that multiplication distributes over addition. The distributive property is also of interest in other mathematical systems.

Definition. Suppose we have a set and two binary operations, $*$ and o, defined on the set. The operation $*$ distributes over the operation o if $a * (b \circ c) = (a * b) \circ (a * c)$ for any elements a, b, c, of the set. (And we can perform all these operations.)

Exercises 12-7

1. Consider the set of counting numbers:

   (a) Is the set closed under addition? under multiplication? Explain.

   (b) Do the commutative and associative properties hold for addition? for multiplication? Give an example of each.

   (c) What is the identity element for addition? for multiplication?

   (d) Is the set of counting numbers closed under subtraction? under division? Explain.

   The answers to (a), (b), and (c) tell us some of the properties of the mathematical system composed of the set of counting numbers and the operations of addition and multiplication.

2. Answer the questions of Problem 1 (a), (b), (c) for the set of whole numbers. Are your answers the same as for the counting numbers?

3. (a) For the system of whole numbers, write 3 number sentences illustrating that multiplication distributes over addition.

   (b) Does addition distribute over multiplication? Try some examples.

4. The two tables below describe a mathematical system composed of the set {A, B, C, D} and the two operations * and o.

*	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

o	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

(a) Do you think * distributes over o? Try several examples.

(b) Do you think o distributes over *? Try several examples.

5. Answer these questions for each of the following systems. Is the set closed under the operation? Is the operation commutative? associative? Is there an identity? What elements have inverses?

   (a) The system whose set is the set of odd numbers and whose operation is multiplication.

   (b) The system whose set is made up of zero and the multiples of 3, and whose operation is multiplication.

   (c) The system whose set is made up of zero and the multiples of 3 and whose operation is addition.

   (d) The system whose set is made up of the rational numbers between 0 and 1, and whose operation is multiplication.

   (e) The system whose set is made up of the even numbers and whose operation is addition. (Zero is an even number.)

   (f) The system whose set is made up of the rational numbers between 0 and 1, and whose operation is addition.

6. (a) In what ways are the systems of 5(b) and 5(c) the same?

   (b) In what ways are the systems of 5(a) and 5(b) different?

*7. Make up a mathematical system of your own that is composed of a set and two operations defined on the set. Make at least partial tables for the operations in your system. List the properties of your system.

*8. Here is a mathematical system composed of a set and two operations defined on that set.

Set: All counting numbers.

Operation *: Find the greatest common factor.

Operation o: Find the least common multiple.

(a) Does the operation * seem to distribute over the operation o? Try several examples.

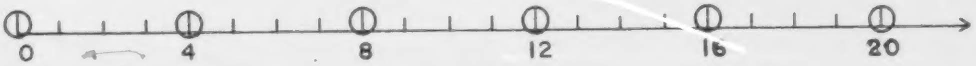
(b) Does the operation o seem to distribute over the operation *? Try several examples.

12-8. Modular Arithmetic

In Section 12-1 we studied a new addition done by rotating the hand of a clock. Using a four-minute clock, we said that $2 + 3 \equiv 1 \pmod{4}$. The tables which we made described the mathematical system mod 4.

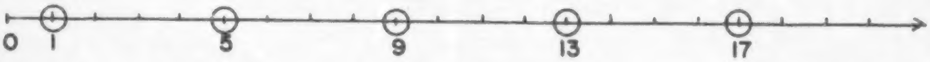
Modular systems are the result of classifying whole numbers in a certain way. For example, we could classify whole numbers as even or odd. In this case, every other number from 0 is put in the same family and the family is named by its smallest member: 0. This is the mod 2 system. Thus the class of all even numbers is 0 (mod 2). Starting from 1, every other number belongs to the same family which we call 1 (mod 2). For the odds and evens, we then have two classes, 0 (mod 2) and 1 (mod 2). The number 5 belongs to the class 1 (mod 2), eight belongs to the class 0 (mod 2).

If we put every fourth number in the same class, we have the mod 4 system. Here is a sketch of some of the numbers belonging to the class 0 (mod 4).



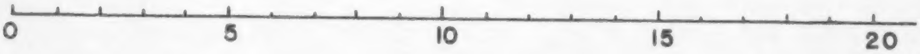
Every fourth number from 0 belongs to the same class. Thus, numbers which are multiples of 4 belong to the class 0 (mod 4).

Here is a sketch showing some of the numbers which belong to the class 1 (mod 4).



Every fourth number from 1 belongs to the same class, that is, 1 (mod 4). Thus the numbers which are 1 plus a multiple of 4 belong to this class.

The two sketches below show respectively some of the numbers which belong to the class 0 (mod 5) and the class 3 (mod 5).



The numbers belonging to the class 0 (mod 5) are multiples of 5.

The numbers belonging to the class 3 (mod 5) are 3 plus multiples of 5.

Our first problems in a modular system used the operation of addition. If we change the operation to multiplication, we get a different mathematical system. With both operations, modular arithmetic is more like ordinary arithmetic than it was with just one operation.

Below are two partly completed modular multiplication tables.

Mod 5

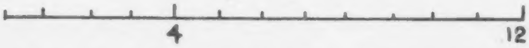
x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2					3
3			1		
4					

Mod 8

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2							4	
3						7		
4								
5								
6								
7								

To find the product 2 × 4 (mod 5), we found the product of 2 and 4 and then found the class to which it belongs. Thus $2 \times 4 = 8$; 8 belongs to the class 3 (mod 5) (eight is the 5th number from 3). Hence $2 \times 4 \equiv 3 \pmod 5$. Also, $3 \times 2 \equiv 1 \pmod 5$. Six is the 5th number from 1.

An example of the computation in the mod 8 table is: $2 \times 6 = 12$; 12 belongs to the class 4 (mod 8). $2 \times 6 \equiv 4 \pmod 8$.



In Section 1, a mathematical system, described by the table, was the result of rotations of the hand of a clock. If we think of the positions as being numbers and the + sign as addition we would have modular addition. Thus, from the above discussion of modular systems, 3 + 2 and 1 are in the same class, mod 4, since 3 + 2 or 5 is the fourth number from 1. Also 3 + 3 and 2 are in the same class, mod 4, since 3 + 3 or 6 is the fourth number from 2.

Exercises 12-8

1. Copy and complete the mod 5 and mod 8 multiplication tables.
2. Answer each of the following questions about the mathematical systems of multiplication mod 5 and mod 8.
   (a) Is the set closed under the operation?
   (b) Is the operation commutative?
   (c) Do you think the operation is associative?
   (d) What is the identity element?
   (e) Which elements have inverses, and what are the pairs of inverse elements?
   (f) Is it true that if a product is zero at least one of the factors is zero?
3. Complete each of the following number sentences to make it a true statement.
   (a) $2 \times 4 \equiv ?\ (\text{mod } 5)$ (c) $5^2 \equiv 1\ (\text{mod } ?)$
   (b) $4 \times 3 \equiv ?\ (\text{mod } 5)$ (d) $2^3 \equiv 0\ (\text{mod } ?)$
4. Find the products:
   (a) $2 \times 3 \equiv ?\ (\text{mod } 4)$ (e) $4^3 \equiv ?\ (\text{mod } 5)$
   (b) $2 \times 3 \equiv ?\ (\text{mod } 6)$ (f) $6^2 \equiv ?\ (\text{mod } 5)$
   (c) $5 \times 8 \equiv ?\ (\text{mod } 7)$ *(g) $6^{256} \equiv ?\ (\text{mod } 5)$
   (d) $3 \times 4 \times 6 \equiv ?\ (\text{mod } 9)$
5. Find the sums:
   (a) $1 + 3 \equiv ?\ (\text{mod } 5)$ (c) $2 + 4 \equiv ?\ (\text{mod } 5)$
   (b) $4 + 3 \equiv ?\ (\text{mod } 5)$ (d) $4 + 4 \equiv ?\ (\text{mod } 5)$
6. (a) Do you think multiplication mod 5 distributes over addition mod 5?
   (b) Do you think addition mod 5 distributes over multiplication mod 5?

Remember that division is defined after we know about multiplication. Thus, in ordinary arithmetic, the question "Six divided by 2 is what?" means, really "Six is obtained by multiplying 2 by what?" An operation that begins with one of the numbers and the "answer" to another binary operation and asks for the other number, is called an inverse operation. Division is the inverse of the multiplication operation.

*7. Find the quotients:

(a) $2 \div 3 \equiv ?$ (mod 8) (e) $0 \div 2 \equiv ?$ (mod 5)
(b) $6 \div 2 \equiv ?$ (mod 8) (f) $0 \div 4 \equiv ?$ (mod 5)
(c) $0 \div 2 \equiv ?$ (mod 8) (g) $7 \div 3 \equiv ?$ (mod 10)
(d) $3 \div 4 \equiv ?$ (mod 5) *(h) $7 \div 6 \equiv ?$ (mod 8)

8. Find the following, remember that subtraction is the inverse operation of addition.

(a) $7 - 3$ (mod 8) (c) $3 - 4$ (mod 8)
(b) $3 - 4$ (mod 5) *(d) $4 - 9$ (mod 12)

9. Make a table for subtraction mod 5. Is the set closed under the operation?

10. Find a replacement for x which will make each of the following number sentences a true statement. Explain.

(a) $2x \equiv 1$ (mod 5) (d) $3x \equiv 0$ (mod 6)
(b) $3x \equiv 1$ (mod 4) (e) $x \cdot x \equiv 1$ (mod 8)
(c) $3x \equiv 0$ (mod 5) (f) $4x \equiv 4$ (mod 8)

11. In problem 10 (d) and (f), find at least one other replacement for x which makes the number sentence a true statement.

12-9. Summary and Review

Summary

A binary operation defined on a set is a rule of combination by means of which any two elements of the set may be combined to determine one definite thing.

A mathematical system is a set together with one or more operations defined on that set.

A set is closed under a binary operation if every two elements of the set can be combined by the operation and the result is always an element of the set.

An identity element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are inverses of each other under a certain binary operation if their combination (in either order) is the identity element for that operation.

A binary operation is commutative if, for any two elements, the same result is obtained by combining them first in one order, and then in the other.

A binary operation is associative if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third. $a + (b + c) = (a + b) + c$.

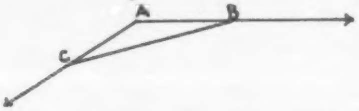
The binary operation * distributes over the binary operation o provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

Sample Review Questions

1. Find the greatest common factor of 8, 12, and 30.
2. The base two numeral for the number twelve has how many digits?
3. What is the intersection of the set of whole numbers between 1 and 11 and the set of even numbers between 5 and 15?
4. Find the simplest numeral for $\frac{16}{25}$ divided by $\frac{8}{5}$.
5. What is the largest number of rays indicated in this diagram?



6. What is the reciprocal for $\frac{2}{9}$ in the set of rational numbers?
7. Explain one way of determining if $\frac{a}{b} = \frac{c}{d}$ is a true statement.
8. If the length of a line segment is measured to the nearest $\frac{1}{2}$ inch, what is the greatest possible error?
9. Find 75% of 14.
10. Find another name for $\frac{7}{8}\cdot\frac{7}{8}\cdot\frac{7}{8}\cdot\frac{7}{8}\cdot\frac{7}{8}\cdot\frac{7}{8}$ using an exponent.
11. If two angles of an isosceles triangle measure 64° and 58°, what is the measure of the third angle?
12. Which of the following measures of central tendency will always be one of the original data? Mean, median, mode.

CHAPTER 13

MATHEMATICS AT WORK IN SCIENCE

13-1. The Scientific Seesaw

Have you ever played on a seesaw?

If your weight is 100 pounds and your partner on the other side of the seesaw weighs 85 pounds, where does he have to sit to make the two sides balance? Will he be closer to the center or farther from it than you? Can you tell how far?

The seesaw is one form of simple machine that is used a great deal around home, at work, and in science laboratories. It belongs to the family of machines called "levers." Of course, not all levers work like seesaws! You use a lever to open a soft drink bottle. You use other forms of levers to jack up your car or to pry up a stone. Can you think of other examples of levers?

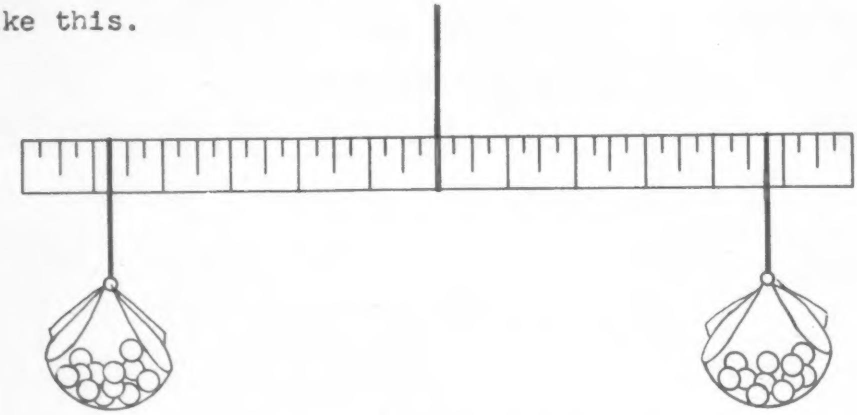
The scientist uses levers of rather fine construction for many purposes in his laboratory. The simplest type is the common laboratory balance or scale. You probably have one in your science room. Scientists long ago studied these scientific seesaws, learned how to balance weights, and expressed their findings in a mathematical formula. Having the formula makes the scientific seesaw much easier to use and understand.

Today you are going to play the part of a scientist. You will set up a simple seesaw experiment, make observations, try to discover a rule and try to state the rule in mathematical form.

The experiment suggests how a scientist makes observations in the laboratory, studies them mathematically and draws conclusions from them. He then tries to state the conclusions by means of a mathematical equation. Finally he uses the formula to predict a new result and then goes back to the laboratory to test whether the rule works in similar situations.

13-2. A Laboratory Experiment

Your equipment to study the scientific seesaw will look something like this.



The materials required in your laboratory are:

A meter stick or yardstick.

Strong thread and two bags to hold the weights; (thin plastic makes very satisfactory bags).

A set of metric weights. (If a set of weights is not available, a supply of pennies or marbles can be used.)

Procedure:

1. Balance the stick by suspending it from a strong thread tied at the middle of the stick. This point at which the stick is suspended is called the fulcrum.

2. Hang ten equal weights on one side of the fulcrum and ten equal weights on the other side and try to make them balance. Take two unequal weights and try to make them balance. Do you find that you must change the distance to achieve a balance when the weights are unequal?

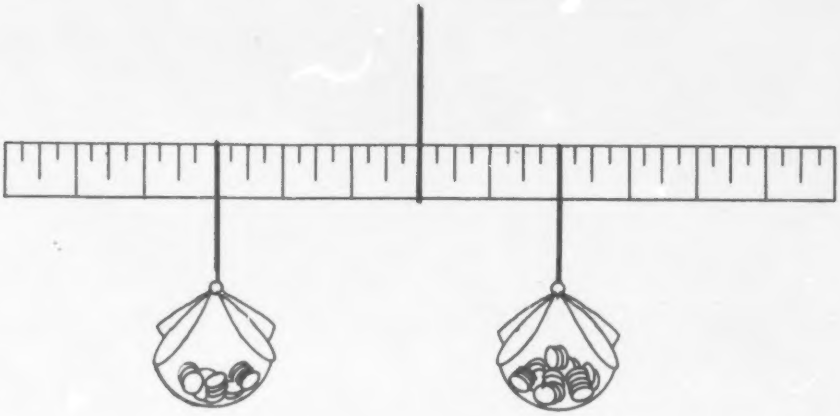
Note: Scientists usually perform some preliminary tests to determine the best way to set up and carry out the experiment. Their first experimental set-up doesn't always work perfectly! You may find it advisable to make some improvements in your equipment and procedures at this stage.

When you have your equipment operating smoothly, you are ready to take the first steps in your experiment. Scientists usually have the experiment carefully planned out in advance, but we shall develop our plan as we go along.

3. (a) Hang a weight of 10 grams (or pennies or marbles if you do not have the metric weights) at a distance of 12 centimeters from the fulcrum, and balance it with a 10 gram weight on the other side. (On a yardstick you may find that $\frac{1}{2}$ inch is a convenient unit of distance.) Observe the distance of this second weight when the lever is in balance and record the distance in a table, column (a), similar to the one below. Note that w and d represent the weight and distance respectively on one side of the fulcrum. W and D represent the weight and distance on the other side of the fulcrum.

TABLE I

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
w =	10	10	10	10	10	10	10	10	10	10
d =	12	12	12	12	12	12	12	12	12	12
W =	10	20	5	8	15	24	12			
D =										



(b) Now double the weight W (make it 20 grams) and find where it must be placed to balance weight w. Write the distance in your table under "20", column (b).

(c) Make weight W only half as large as it was in the first case, adjust the balance and read the distance from the fulcrum. Write it in the table, column (c) under "5".

(d) Notice that in these first three trials, weight w and its distance d from the fulcrum remained the same. All changes were made in weight W and balanced by changes in D. Use the weights indicated in columns (d) - (g) and find the value of D. Make several other changes for W and write the corresponding results for D in your table--columns (h), (i), and (j).

4. Now, as indicated in Table II, let weight w be 16 grams and its distance from the fulcrum 6 cm. Find how heavy weight w will have to be to balance at 6 cm. on the other side of the fulcrum. What weight will balance the lever 4 cm. from the fulcrum? 16 cm? Try several other distances from the fulcrum, find the weight which will just balance the lever, and fill in your Table II.

TABLE II

w	16	16	16	16	16	16	16
d	6	6	6	6	6	6	6
W							
D	6	4	16				

5. Try other weights and distances as suggested in Tables III and IV and fill in similar tables of your own.

TABLE III

w	20	40	10			
d						
W	20	20	20	20	20	20
D	15	15	15	15	15	15

TABLE IV

w	18	18	18	18	18	18	18
d	5	5	5	5	5	5	5
W	15	10	30	45			
D					10	15	18

13-3. Caution: Inductive Reasoning at Work!

After a scientist has completed an experiment and collected data, he tries to analyze it. He tries to discover all the facts that the data present. He attempts to interpret these facts in a connected and convenient way and to discover a general rule which the data suggest. His hope is to state these results in a precise way, preferably by a simple mathematical formula.

Note that the scientist studies a number of specific experimental results and from these tries to reach a conclusion which will hold for all cases. This is the kind of inductive reasoning which we discussed earlier. It should be used with caution. There may well be occasions when a few examples suggest a conclusion which is not true in general. Thus, if you go to New York and meet five people in succession with red hair, it is not safe to conclude that everyone in New York has red hair. Also, if you notice that $\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$, $\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$, $\frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8}$, and $\frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}$ you will be in trouble if you assume that you may always cross off numerals in this way.

When a general rule has been suggested, the scientist tries to verify it by further experiment and, if possible, by deductive reasoning. In all these steps mathematics and mathematical reasoning are especially important.

Let us look at the results in our tables from this point of view. We are trying to determine whether there seems to be a general rule which describes all these relationships. If possible, we wish to express the rule in mathematical terms. If it is a general law, we should be able to use it to predict where to place

Exercises 13-3

1. a. In Table I, do you notice any connection between the location of equal weights on opposite sides of the fulcrum? What is it?

   b. If weight W is doubled, w and d remaining unchanged, how does its distance D from the fulcrum change?

   c. If weight W is made half as much, how does its distance D from the fulcrum change?

   d. Do the values for w, d and W, D appear to be related in any way? Can you state a general rule that seems to hold concerning w, d, W, and D? State the rule in words and then in the form of a mathematical equation in which you use the symbols w, W, d, and D.

   e. Check your rule by applying it to some of the entries found by experiment in Tables II, III, IV.

2. Use the equation suggested in the preceding exercise to predict the missing entries in Table V.

TABLE V

w	25	35	12	45	21	15	23	11		14	10	100	100	100
d	3	4	5	3	5		4	8	7	5.5	50	50	500	5000
W	15		7		14	20	12		13	10				
D		7		9		4.5		10	8		5	5	5	5

3. Go back to your experiment and check the results in Table V to see if they actually produce a balance.

13-4. Graphical Interpretation

In Chapter 11 you learned about graphs and their usefulness in presenting numerical information in a clear and condensed way. Scientists often make a graph of the observations obtained through experimentation for the help it can give them in summarizing and interpreting the data.

In the equation $wd = WD$ which you obtained for yourself in the preceding experiment, four quantities are involved. This equation can be interpreted in a number of ways corresponding to the way you followed through in the experiment. In the first part of the experiment, you chose fixed values for w and d and then found the values for W and D which produced a balance. From each experiment you got one pair of values which satisfied the relation $WD = 120$. A graph of $WD = 120$ pictures all the pairs of values which produce a balance when $wd = 120$. The graph, then, supplies not only all the information in Table I but other possible values for W and D.

The next step is to draw the graph of the relation between weights (W) and distances (D) in Table I.

If you need help in drawing the graph, the following suggestions should aid you:

Use graph paper and begin with two perpendicular lines called axes. The intersection of the axes is named point O.

Label the horizontal axis W and the vertical axis D.

If you use $\frac{1}{4}$ inch squared paper, a suitable scale is one for each space.

In Table I, the first weight is 10 and the corresponding distance is 12. Locate 10 on the W axis. Follow the vertical line through 12 on the D axis. This point is called (10, 12). Make a small dot for the point.

Similarly, from Table I, when w = 20, D = 6. Locate 20 on the W axis. Follow the vertical line through 20 to the point where it meets the horizontal line through 6 on the D axis. This point is called (20, 6).

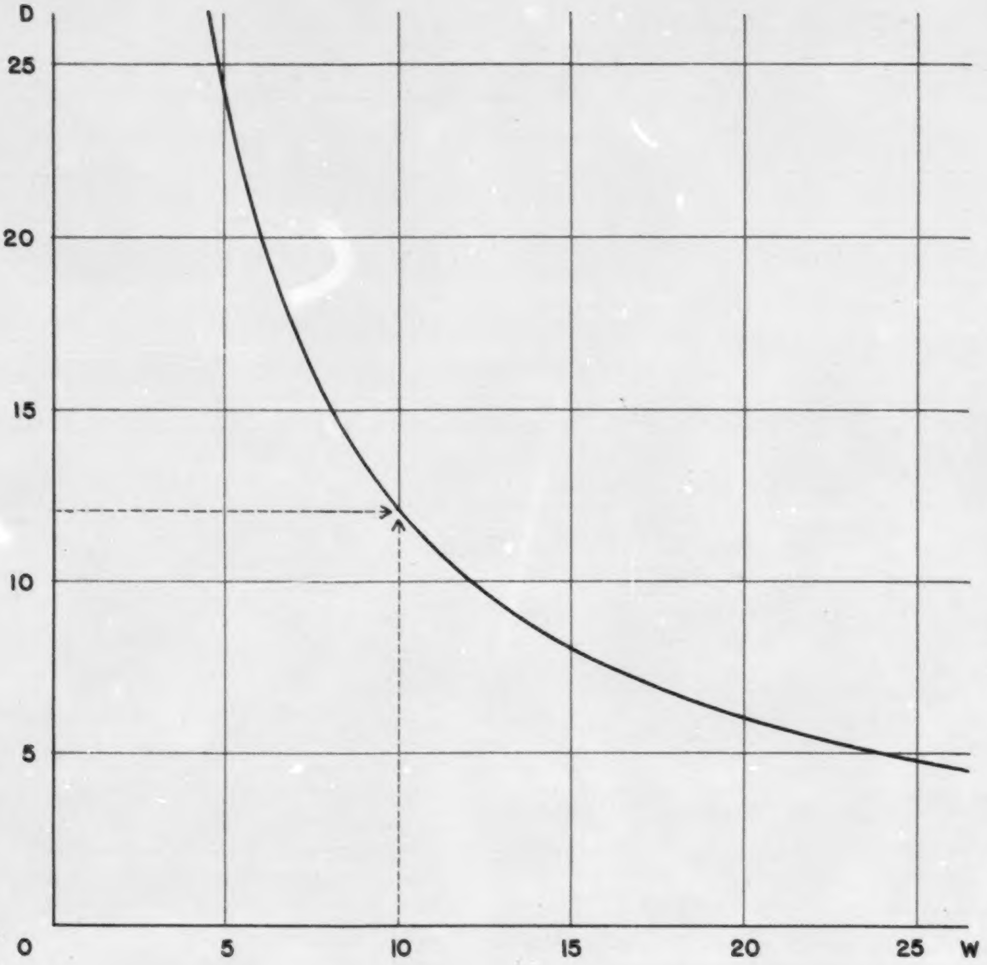
Before you read on, locate and mark the other points from Table I. These are (5, 24), (8, 15), (24, 5), (12,10).

Mark the points which correspond to the results you found in columns (h), (i), and (j) in Table I.

Fill in the blanks in the following pairs for (W, D) and mark the corresponding points: (4,); (, 6); (16,); (, 18). Use WD = 120.

Draw a smooth freehand curve through the points you have located. This curve gives you the general picture of the relation between weights and distances as in Table I. If any point seems to lie to one side or the other of your smooth curve, check your computation. Not all experiments turn out perfectly and not all results fall into neat patterns at once. The points obtained from the measurements you made should fall near the curve. Often scientists expect no more than this from an experiment of this type.

The curve that you have drawn is a portion of a curve called a hyperbola. You will learn more about this curve in your study of algebra.



GRAPH OF WD = 120
FROM DATA IN TABLE I

Exercises 13-4

1. Study the graph and then answer these questions:

   (a) What happens to the distance as the weight increases?

   (b) What happens to the weight as the distance increases?

2. In order to find the value of W when D is 24, locate 24 on the axis and follow the horizontal line through 24 until it meets the curve. Read the value on the W scale directly beneath this point. You should have 5. Find the values of W from the graph for the following points:

   (a) (, 18) (b) (, 15) (c) (, 9)

3. Find the values of D from the graph:

   (a) (6,) (b) (4,) (c) (15,)

4. Tell whether or not each of the following points is on the graph:

   (a) (10, 25) (b) (15, 8) (c) (5, 5) (d) (20, 15)

5. Estimate the missing values:

   (a) (7,) (b) (8,) (c) (, 9) (d) (17,)

   (e) (, 21) (f) (23,)

6. Draw a graph of the relation between W and D given in Table II. Use the formula WD = 96 to find the number pairs you need for locating points. Check the values you find with those in Table II.

7. From the graph find D when W is 20; find W when D is 12.

8. What happens to the distance when the weight decreases? increases?

9. Does this graph have anything in common with the graph you drew for WD = 120?

13-5. Other Kinds of Levers

In the introduction to this chapter you read that not all levers are like seesaws. You may be interested in asking about this in your science class at an appropriate time. In addition to the automobile jack, bottle opener, and crowbar mentioned before, there are other examples of widely used forms of the lever. You may think of ice tongs, nutcrackers, scissors, claw hammers, pliers, pruning shears, and hedge clippers as some useful levers.

13-6. The Role of Mathematics in Scientific Experiment

Although the experiment using the lever does not use a great deal of mathematics, it does suggest how mathematics is used in scientific activities. You saw how mathematics was used in measuring, counting, and comparing quantities. You noted how observations of data were recorded in mathematical terms.

You searched for a pattern by studying the numbers in your recorded data. By reasoning from a set of specific cases you developed a general statement to be applied in all similar situations. This kind of reasoning is called inductive reasoning. It leads from a necessarily restricted number of cases to a prediction of a general relationship. This general relationship was stated in mathematical symbols in an equation: $WD = wd$. To establish this general principle further experimentation was performed.

In addition, you drew a graph of $WD = 120$ and of $WD = 96$ to show how these statements tell the complete story in each case. The graph is another instance of the use of mathematics to interpret and to summarize a collection of facts. The graph also helped to reveal the general pattern which was discovered.

Many scientific facts were undiscovered for thousands of years until alert scientists carefully set up experiments much as you have done and made discoveries on the basis of observations. Some examples of these are the following:

(a) For thousands of years, people assumed that if a heavy object and a light object were dropped at the same time, the heavy one would fall much faster than the light one. Look up the story of Galileo and his experiment with falling objects and see what he discovered.

(b) From time immemorial, people watched eclipses of the sun and moon and saw the round shadow of the earth but did not discover that the earth was round. Eratosthenes, in 230 B.C., computed the distance around the world by his observations of the sun in two locations in Egypt, yet seventeen hundred years later when Columbus started on his journey, many people still believed the world was flat. Look up in a history of mathematics book or in an encyclopedia the story of Eratosthenes and this experiment.

(c) People had watched pendulums for many centuries before Galileo did some measuring and calculating and discovered the law which gives the relation between the length of the pendulum and the time of its swing.
Look up this experiment in a history of mathematics book.

Notice that all these experiments are based on many careful measurements and observations in order to discover the scientific law. Then the law is stated in mathematical terms. A great deal of science depends upon mathematics in just this way.

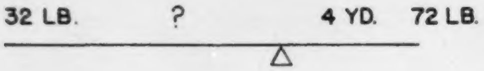
The examples which we have given here describe older fundamental discoveries all of which used relatively simple mathematics. The scientists of today are using more advanced mathematics, and many of the newer kinds of mathematics, in their scientific experiments.

Exercises 13-6

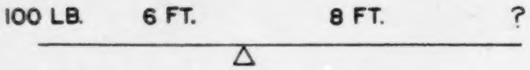
1. Some seesaws are fitted so that they can be shifted on the support. Why?

Find the missing value in each case:

2.



3.



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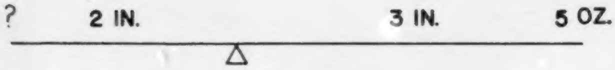

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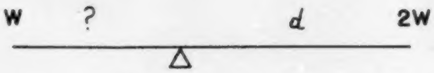

6. Do you suppose a 90 lb. girl could ever lift 1000 pounds? Justify your answer.

7. A child who weighed 54 pounds asked his father (180 pounds) to ride a seesaw with him. The seat for the child was 6 feet from the fulcrum. Where should the father sit to balance the child?

8. An iron bar 3 feet long is to be used to lift a weight of 75 pounds. The fulcrum is 6 inches from the weight. A force of

how many pounds is needed to lift the weight?

9. Suppose a 100 pound boy sits on one end of a six foot crowbar, $5\frac{1}{2}$ feet from the fulcrum. How heavy a weight can he lift at the other end of the bar?

10. Which points lie on the graph of $WD = 60$?

(a) $(12, 5)$ (b) $(\frac{1}{3}, 180)$ (c) $(1, 6)$

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(Galileo, pp. 726-770; falling bodies, p. 729; pendulum, pp. 729, 741; Eratosthenes, pp. 205-207.)

2. Kline, Morris. MATHEMATICS IN WESTERN CULTURE. New York: Oxford University Press, 1953.

Chapter XIII The Quantitative Approach to Nature
Chapter XIV The Deduction of Universal Laws